On well-posedness and instability for *steady* temperature-dependent shallow ice flow

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22 March, 2007 (SIAM GS 07 Santa Fe)

Outline

A good place to start the study of the nonlinear system $\dot{x} = f(x)$ is by finding the *zeros* of f or the *fixed points* of $\dot{x} = f(x)$. These are also referred to as *zeros*, *equilibria*, or *stationary solutions*. Even this may be a formidable task . . .

from (Guckenheimer & Holmes 1983)

Acknowledgements. Jed Brown, Craig Lingle, and I developed PISM. Christian Schoof helped me solidify my spokey thoughts.

Ice flow models generally

We need to agree on meaning of "ice flow model"

- "Ice flow models" are for glaciers, ice sheets, ice streams, and floating ice shelves, but not for (e.g.) sea ice or uncompacted snow or for (e.g.) the propagation of waves elastically in ice.
- Ice flow is a gravity-driven slow (viscous) flow (because force of inertia adds almost nothing to the force balance).
- Ice flow is a continuum process; no crevasses or detailed calving.
- There should be a conservation of energy equation because flow can depend strongly on temperature and must incorporate strain-dissipation heating.

Such models are Stokes flows which include non-Newtonian and temperature-dependent flow.

(There is wide agreement that such models should suffice. At this level of generality, I *think* such models are not on trial.)

For simplicity in this talk . . .

Let's only consider

- isotropic ice flow
- cold (and not polythermal) ice flow
- nonsliding ice sheets

Such a model is credible for:

Small polar ice field on Axel Heiberg Island, Nunavut, Canada. Photo 119, Post & LaChapelle 2000.

The thermo-coupled shallow ice approximation

The mathematical model on trial is also shallow.

Derivation: Aspect ratio of $\epsilon = d/L$. Suppose pressure is hydrostatic plus size ϵ; make other shallow scaling assumptions like this. Rewrite the Stokes problem and conservation of energy in nondimensional variables. Only ϵ^0 and ϵ^2 terms appear; drop the ϵ^2 terms.

The thermo-coupled shallow ice approximation, cont.

For the resulting model (equations on next slide):

- only stresses left in the force balance are "vertical" shears
- velocities are determined by geometry of ice and temperature field within ice
- no heat conduction in the horizontal
- 2D problem for ice surface; 3D problem for temperature

This is the

thermomechanically coupled shallow ice approximation (SIA).

Admittedly rhetorical question: Is it a good model for ice sheets?

Equations of steady thermocoupled SIA

Assume flat bed and no basal sliding. Let $R = \{0 < z < H(x,y)\}$ be 3D domain of ice. Assume temperature-dependent Glen constitutive relation $\dot\epsilon_{ij} = A(T)\,(\sigma')^{n-1}\,\sigma'_{ij}$. Note shallowness gives pressure $P = \rho g H$ and $\sigma' = P|\nabla H|$. On R we have incompressible fluid with velocity $\mathbf{v} = (v_1, v_2, v_3)$ and equations

$$\textit{energy conservation}: \quad \mathbf{v}\cdot\nabla T = K\frac{\partial^2 T}{\partial z^2} + \Sigma(H,T)$$

$$\textit{horizontal velocity}: \quad (v_1, v_2) = -2\rho g \nabla H (P|\nabla H|)^{n-1}\int_0^z A(T)(H-\zeta)\,d\zeta$$

To determine $H(x,y)$ we have

$$\text{2D } \textit{mass conservation}: \quad 0 = a - \nabla\cdot\left(\int_0^H (v_1, v_2)\,dz\right)$$

The free boundary (margin) problem for H has $H \geq 0$ as constraint.

Boundary conditions for T on top and bottom of R: $T = T_s$ when $z = H$ and $\partial T/\partial z = -(\gamma/k)$ when $z = 0$.

EISMINT II (2000) = standardized "easy" problems

- Let $\Omega = [-750\text{km}, 750\text{km}] \times [-750\text{km}, 750\text{km}] \subset \mathbb{R}^2$.
- Suppose a *flat base* and a constant geothermal flux rate at base.
- Suppose a steady surface temperature

$$T_s(r) = T_{\text{min}} + C\,r.$$

- Suppose a well-behaved steady accumulation function $a(r)$ which has $a(0) > 0$ (accumulation at center) and $a(r) < 0$ when $r > 500$ km.
- Start with zero thickness and compute thickness and temperature at end of a $200,000$ year run.

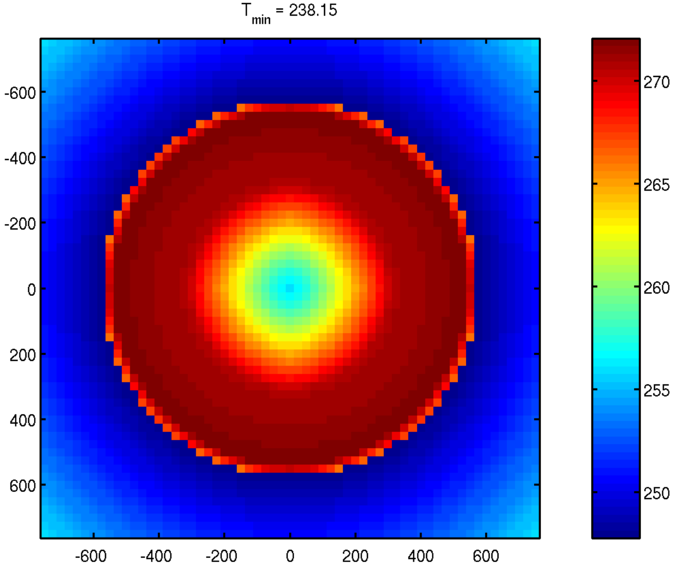
This continuum problem is EISMINT II experiment A or F (*depending on* T_{min} *only*). It has perfect angular symmetry.

A movie ...

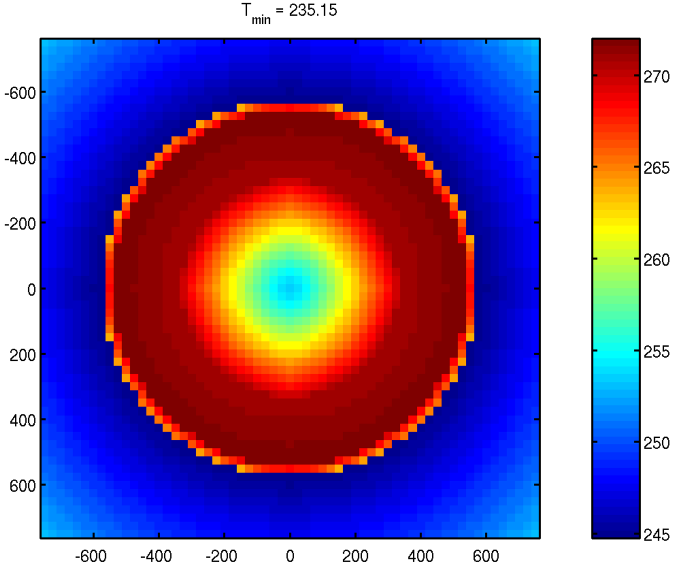
The next few slides show basal temperatures from PISM on a $61 \times 61 \times 201$ uniform cartesian grid at end of $200,000$ year run.

(PISM = Parallel Ice Sheet Model. PISM uses low-order finite difference schemes. PISM's approximation has been better verified than other ice sheet models at this point.)

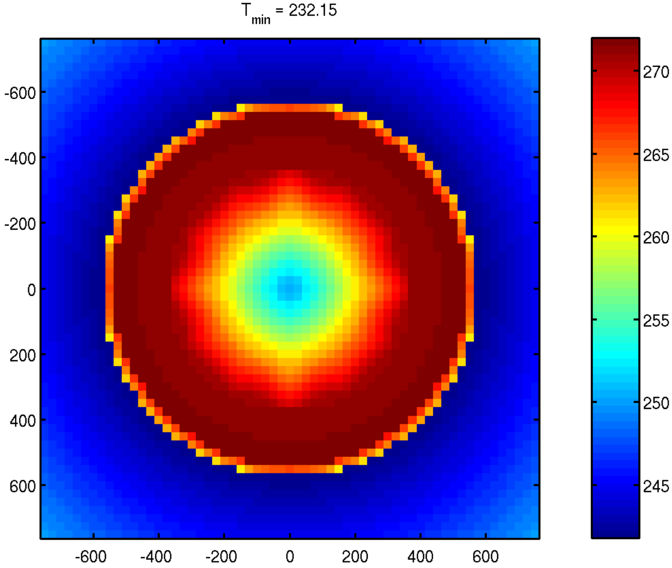
EISMINT II example: $T_{\text{min}} = 238.15$ (experiment A)



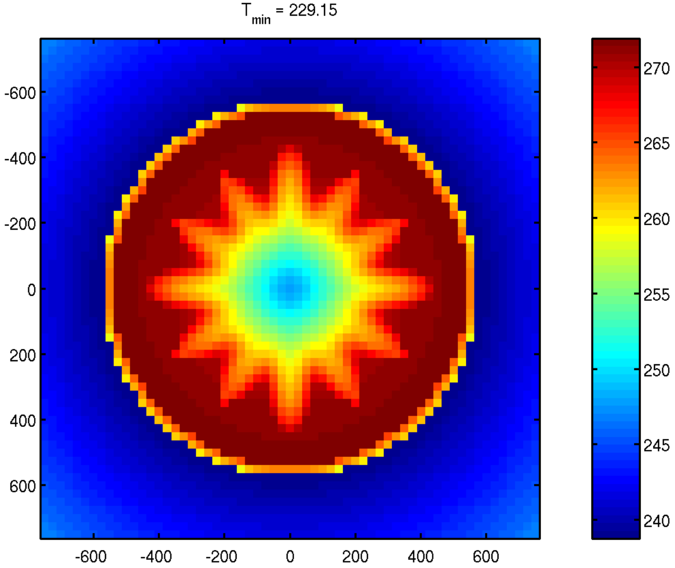
EISMINT II example: $T_{\text{min}} = 235.15$



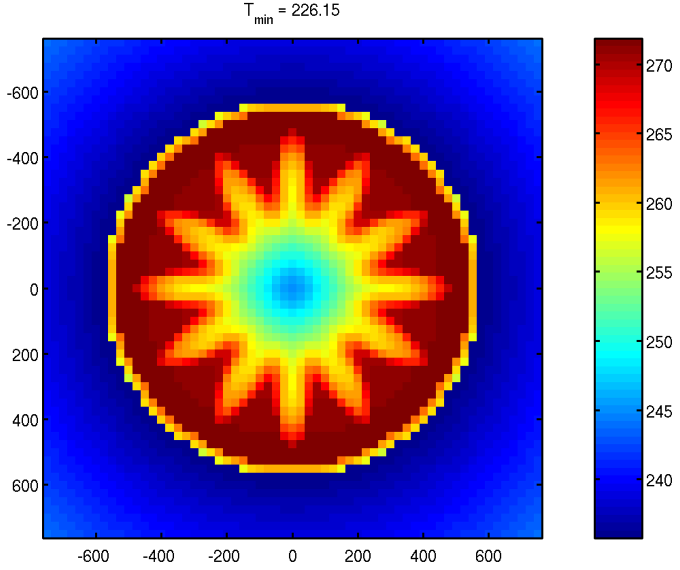
EISMINT II example: $T_{\text{min}} = 232.15$



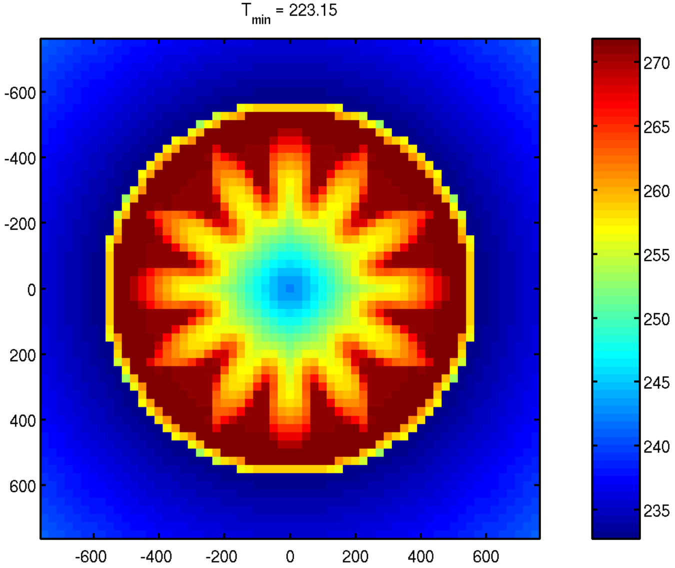
EISMINT II example: $T_{\text{min}} = 229.15$



EISMINT II example: $T_{\text{min}} = 226.15$



EISMINT II example: $T_{\text{min}} = 223.15$ (experiment F)



Interpretations

- Emergent ice streams! (Payne)
- Ill-posedness! (Hindmarsh)
- Numerical error! (Bueler)
- Unstable equilibrium point! (Schoof)

These are serious people. I feel bad caricaturing their ideas. I think there is an element of truth in all of the above interpretations. See Bueler, Brown, Lingle, Exact solutions to the thermomechanically coupled shallow ice approximation, *to appear* J. Glaciol.

Note that EISMINT II experiments A and F are really steady state computations. There is no particular significance to prescribed run of $200{,}000$ years except "approximate equilibrium is achieved".

So here's a different question re spokes: How is the configuration of a steady ice sheet determined by a steady climate?

A map from climate to steady ice sheet

What is the map

$$\mathcal{C}: \quad \text{climate} \quad \to \quad \text{ice sheet(s)}$$

if climate is essentially constant for long enough for the ice sheet (or multiple ice sheets, e.g. globally) to achieve approximate equilibrium?

Obviously this map relates to the $t \to \infty$ limit of the dynamical ice sheet model if climate inputs are held fixed. Note that there could be a limiting periodic orbit or strange attractor and thus $\mathcal{C}$ might not make sense. But recall Guckenheimer & Holmes advice ...

Goal of this talk: describe a mathematical model in which one can address the steady-climate-to-steady-ice-sheet map directly and precisely.

Steady climate input

For this talk, steady climate consists of time-independent functions of horizontal position for *accumulation* $a(x, y)$, *bed elevation* $b(x, y)$, *ice surface temperature* $T_s(x, y)$, and *geothermal flux input to the ice base* $\gamma(x, y)$.

We think of steady climate as parameters μ to dynamical ice sheet system $\dot{x} = f_\mu(x)$ and consider stationary point solutions to $0 = f_\mu(x)$.

[Note that for real ice sheets there are *complications* (e.g. elevation-dependent accumulation) and *additional climate inputs* (e.g. melt rates under floating ice shelves).]

Map: steady climate $\to$ steady ice sheet

An ice sheet is described by ice thickness $H(x, y)$ and ice temperature $T(x, y, z)$. So

$$\mathcal{C} : (a, b, T_s, \gamma) \mapsto (H, T)$$

is map from steady climate to steady ice sheet.

Questions for time-dependent ice sheet dynamical system. Are limit cycles or strange attractors possible as $t \to \infty$ limit? Where are stationary points? Are they unstable or stable?

Questions about $\mathcal{C}$. When is it well-defined (either single-valued or multi-valued)? Is it continuous?

Major technical issue. The dynamical (time-dependent) ice sheet evolution is described by a *free boundary problem* for the conservation of mass PDE.

Transformed thickness

Calvo and others (2002) noted

$$u = H^{(2n+2)/n} = H^{8/3}$$

much better behaved at margin; Glen exponent $n = 3$ in flow law.

Note isothermal time-dependent SIA is single equation

$$\frac{\partial H}{\partial t} = a + \nabla \cdot \left(\Gamma H^5 |\nabla H|^2 \nabla H\right).$$

Transforms to

$$\frac{\partial (u^{3/8})}{\partial t} = a + \nabla \cdot \left(\tilde{\Gamma} |\nabla u|^2 \nabla u\right).$$

This is a p-Laplacian equation with weird time-dependence; $p = n + 1 = 4$.

Classical (Laplacian) obstacle problem

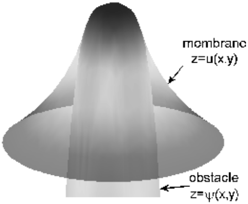
Given *obstacle* $\psi(x,y)$. Let $\mathcal{K} = \{w \ge \psi\} \subset W_0^{1,2}(\Omega)$ be *admissible functions*. Solve

$$\int_\Omega \nabla u \cdot \nabla(w-u) \ge \int_\Omega f(w-u) \qquad \forall w \in \mathcal{K}$$

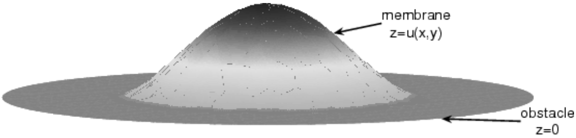
for $u \in \mathcal{K}$. Note that where $u > \psi$ we have

$$-\triangle u = f$$

Two cases where Ω is disc:



$f = 0$



$f > 0$ in central area, $f < 0$ near $\partial\Omega$, and $\psi = 0$

The obstacle for shallow ice sheets

Ice sheet thickness is nonnegative.

Note $H \geq 0$ iff $u = H^{(2n+2)/n} \geq 0$.

Thus

$$\psi(x, y) = 0$$

and

$$\mathcal{K} = \{u \geq 0\} \subset W_0^{1,p}$$

where $p = n + 1$.

μ-weighted p-Laplacian obstacle problem

A slightly-generalized Laplacian obstacle problem is useful for ice sheets:

Let $p > 2$. Let $\Omega \subset \mathbb{R}^2$. Suppose $0 < \mu_1 \le \mu(x,y) \le \mu_2$. Consider

$$\int_\Omega \mu(x,y) |\nabla u|^{p-2} \nabla u \cdot \nabla(w-u) \ge \int_\Omega a(w-u) \qquad \forall w \in \mathcal{K}$$

Theorem. Above variational inequality has unique continuous solution $u \in \mathcal{K}$. Solution has *a priori* bound

$$\|u\|_{W^{1,p}} \le C(p, \mu_1, \mu_2, \|a\|_{L^q}).$$

For isothermal SIA ice sheets on flat beds this shows existence and uniqueness of solution to steady problem.

Equations of steady thermocoupled SIA (again)

Assume flat bed and no basal sliding. Let $R = \{0 < z < H(x,y)\}$ be 3D domain of ice. Assume temperature-dependent Glen constitutive relation $\dot\epsilon_{ij} = A(T)\,(\sigma')^{n-1}\,\sigma'_{ij}$. Note shallowness gives pressure $P = \rho g H$ and $\sigma' = P|\nabla H|$. On R we have incompressible fluid with velocity $\mathbf{v} = (v_1, v_2, v_3)$ and equations

$$\textit{energy conservation}: \quad \mathbf{v}\cdot\nabla T = K\frac{\partial^2 T}{\partial z^2} + \Sigma(H,T)$$

$$\textit{horizontal velocity}: \quad (v_1, v_2) = -2\rho g \nabla H (P|\nabla H|)^{n-1} \boxed{\int_0^z A(T)(H-\zeta)\,d\zeta}$$

To determine $H(x,y)$ we have

$$\text{2D } \textit{mass conservation}: \quad 0 = a - \nabla\cdot\left(\int_0^H (v_1, v_2)\,dz\right)$$

The free boundary (margin) problem for H has $H \ge 0$ as constraint.

Boundary conditions for T on top and bottom of R: $T = T_s$ when $z = H$ and $\partial T/\partial z = -(\gamma/k)$ when $z = 0$.

Obstacle problem for temperature-dependent ice sheets

Suppose a temperature field T everywhere in $\Omega \times [0,\infty)$. Let $n = 3$ for simplicity. Transform thickness $u = H^{8/3}$. Transform above horizontal velocity equation; define

$$\mu(u,T) = 2(\rho g)^3 \, u^{-15/8} \int_0^{u^{3/8}} A(T(s)) \left(u^{3/8} - s\right)^4 \, ds$$

Consider variational inequality:

$$\int_\Omega \mu(u,T) |\nabla u|^2 \nabla u \cdot \nabla (w - u) \geq \int_\Omega a(w-u) \qquad \forall w \in \mathcal{K}$$

Theorem. If $T = T(x,y)$ satisfies $T_1 \le T \le T_2$ then above variational inequality has unique continuous solution $u \in \mathcal{K}$. Solution has *apriori* bound $\|u\|_{W^{1,4}} \le C(T_1, T_2, \|a\|_{L^{4/3}})$.

But $T = T(x,y)$ is a glaciologically silly assumption.

Obstacle problem for temperature-dependent sheets cont.

Conjecture I. Suppose a temperature field $T = T(x, y, z)$ everywhere in $\Omega \times [0, \infty)$. Let $n = 3$. Transform thickness $u = H^{8/3}$. Define $\mu(u, T)$ as above. The variational inequality

$$\int_\Omega \mu(u,T)|\nabla u|^2 \nabla u \cdot \nabla(w-u) \geq \int_\Omega a(w-u) \qquad \forall w \in \mathcal{K}$$

has a unique continuous solution. There is a reasonable an apriori bound.

A mathematical model for steady thermocoupled SIA ice sheets

In above, we are given temperature field and solve for thickness (ice geometry):

$$\mathcal{A} : T \mapsto H$$

Given thickness, one may find velocities by integration and then one may solve conservation of energy:

$$\mathcal{B} : H \mapsto T$$

Conjecture II. Under reasonable assumptions, the regularity of R is good enough so that $\mathcal{B}$ is well-defined. Furthermore, under reasonable assumptions the map

$$\mathcal{N} = \mathcal{A} \circ \mathcal{B} : H \mapsto H$$

has a fixed point in $\mathcal{K} \subset W_0^{1,p}(\Omega)$.

A mathematical model cont.

The (time-dependent) dynamical system for thermocoupled SIA ice sheets will have solutions of above problem as stationary points. Once we can find these points, we can ask about the local dynamical behavior around these points.

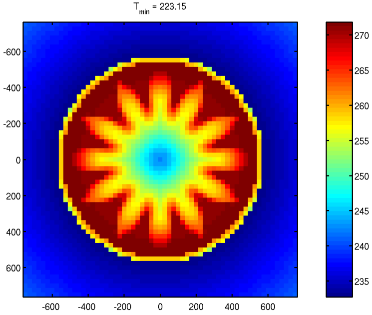
A fixed point of $\mathcal{N}$ could be a stable point for the dynamical ice sheet model, with respect to perturbations. Or not.

Conjecture III. The fixed point corresponding to the exact solution to EISMINT II experiment A *is stable*. The one for EISMINT II experiment F *is not stable*. The experiment F fixed point *is stable with respect to* $SO(2)^\top$ *perturbations*.

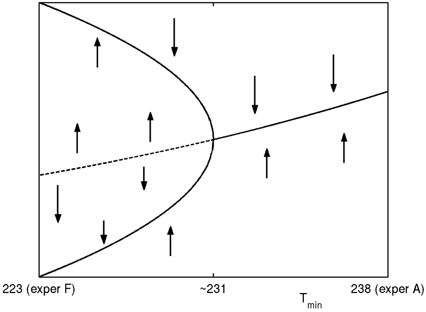
[*Going way out on a limb*: I suspect something like this last conjectured situation *also holds for the non-shallow 3D thermocoupled Stokes equations* applied to an EISMINT II-like problem.]

Remember the movie?

Recall "movie" of EISMINT II basal temperatures. Perhaps we are seeing a bifurcation of the angularly-symmetric exact solution as the single parameter T_{min} drops from 238.15 (A) to 223.15 (F).



$T_{\text{min}} = 223.15$ (experiment F) case



schematic of bifurcation
[*COMMENT ON y-AXIS MEANING*]

The steady-climate-to-steady-ice-sheet map

Recall $\mathcal{C} : (a, b, T_s, \gamma) \mapsto (H, T)$.

- Numerical results of EISMINT II experiment A ($T_{\text{min}} = 238.15$) suggest $\mathcal{C}$ is stable with respect to perturbations of climate input at experiment A. Presumably $\mathcal{C}$ is continuous at experiment A.
- EISMINT II experiment F ($T_{\text{min}} = 223.15$) suggests $\mathcal{C}$ is not continuous at experiment F. That is, for T_{min} less than the bifurcation value, a perturbation to the climate inputs causes the exact angularly-symmetric stationary solution to jump a non-zero distance (if there is a angle-dependent component to perturbation).

One could call non-continuity of $\mathcal{C}$ "ill-posedness" because of "sensitive dependence" of $t \to \infty$ solution of dynamical problem "with respect to inputs". *Probably shouldn't use language that way ...*

Numerical errors get in the way of drawing any conclusions in this area! *Good numerical analysis needed.*

Conclusion

This talk has not really been about a particular problem or model. It is almost a cry for help:

- we need to have decent mathematical models for the steady configuration of glaciers, ice sheets, ice streams, and ice shelves
- such models are mostly not present in the literature; usually only the time-dependent problem (or Stokes w. fixed geometry)
- time-dependent ice sheets are modelled by continuum (nonlinear) dynamical systems, and we may talk about them as such
- equilibrium points of these dynamical systems are the steady ice sheets in question; these can be stable or unstable equilibrium points
- most knowledge of these dynamical systems will come from numerical approximations of PDEs (and variational inequalities); need to use language which addresses the dynamical system while acknowledging ever-present numerical error