Geophysical Ice Flows: Analytical and Numerical Approaches

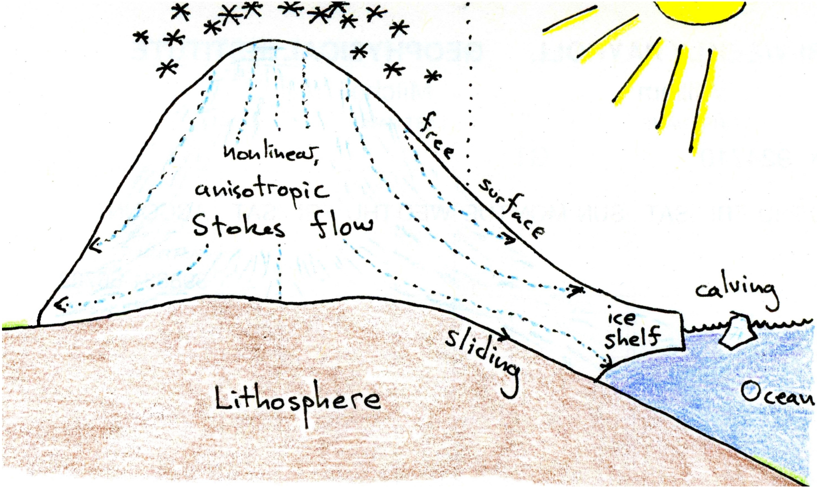
Will Mitchell

University of Alaska - Fairbanks

July 23, 2012

Supported by NASA grant NNX09AJ38G

Ice: an awesome problem



...velocity, pressure, temperature, free surface all evolve

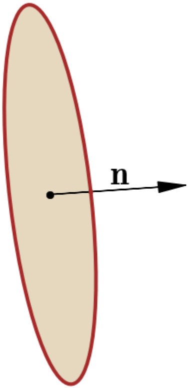
Outline

- I. Introduction to viscous fluids
- II. Exact solutions
- III. Finite element solutions

Stress: force per unit area

A tornado sucks up a penny. At any time:

- The fluid into which $\mathbf{n}$ points exerts a force on the penny
- Force / area = stress
- The stress vector is a linear function of $\mathbf{n}$
- In a Cartesian system: stress $= \sigma \cdot \mathbf{n}$
- σ is the *Cauchy stress tensor*



Quiz:
Suppose there is no $p \geq 0$ such that $\sigma \cdot \mathbf{n} = -p\mathbf{n}$.
Physical interpretation?

Decomposition of stress

- In a fluid at rest, $\sigma \cdot \mathbf{n} = -p\mathbf{n}$, so

$$\sigma = -pI$$

- In general, choose $p = -\mathrm{Trace}(\sigma)/d$, so

$$\sigma = -pI + \tau$$

  where τ has zero trace.

… this defines *pressure* p and *deviatoric stress* τ

Strain rate

- let $\mathbf{u}$ be a velocity field
- the gradient of a vector is the tensor $(\nabla \mathbf{u})_{ij} = \dfrac{\partial u_j}{\partial x_i}$
- define $Du = \dfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$
- in 2D: $Du = \frac{1}{2} \begin{bmatrix} 2\dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} & 2\dfrac{\partial u_2}{\partial x_2} \end{bmatrix}$
- Du is the *strain rate* tensor.

True or False:
"Since Du is a derivative of velocity, it measures acceleration."

Constitutive Laws: Newtonian

How does a fluid respond to a given stress?

- For *Newtonian* fluids (*e.g.* water) a linear law:

$$\tau = 2\mu Du$$

The proportionality constant μ is the *viscosity*.

Constitutive Laws: Glen's

For glacier ice, a nonlinear law.

- define

$$\|\tau\| = \sqrt{\frac{1}{2}\mathrm{Tr}(\tau^T\tau)} \quad \text{and} \quad \|Du\| = \sqrt{\frac{1}{2}\mathrm{Tr}(Du^T Du)}$$

- assume

$$\|Du\| = A\,\|\tau\|^n$$

- the law is either of

$$\tau = (A\,\|\tau\|^{n-1})^{-1} Du$$
$$\tau = A^{-1/n}\,\|Du\|^{(1-n)/n}\,Du$$

A is the *ice softness*, $n \approx 3$ is Glen's exponent.

Stokes Equation

What forces act on a blob occupying a region Ω within a fluid?

- body force, gravity: $\int_\Omega \rho \mathbf{g}$
- force exerted by surrounding fluid: $\int_{\partial\Omega} \sigma \cdot \mathbf{n} = \int_\Omega \nabla \cdot \sigma$

Force = rate of change of momentum

$$\begin{aligned} \int_\Omega \rho \mathbf{g} + \nabla \cdot \sigma &= \frac{\partial}{\partial t} \int_\Omega \rho \mathbf{u} \\ &= \int_\Omega \frac{D}{Dt} \rho \mathbf{u}. \end{aligned}$$

In glaciers, $Fr = \left| \frac{D}{Dt} \rho \mathbf{u} \right| : |p| < 10^{-15}$ so

$$\rho \mathbf{g} + \nabla \cdot \sigma = \mathbf{0}.$$

Incompressible Stokes System (two versions)

$$\begin{cases} \rho\mathbf{g} + \nabla \cdot \sigma & = \mathbf{0} \\ \nabla \cdot \mathbf{u} & = 0 \end{cases}$$

$$\begin{aligned} \nabla \cdot \sigma &= \nabla \cdot \tau + \nabla p \\ &= 2\mu\nabla \cdot \dot{\epsilon} - \nabla p \\ &= \mu\nabla \cdot (\nabla \mathbf{u}) + \mu\nabla \cdot (\nabla \mathbf{u}^T) - \nabla p \end{aligned}$$

$$\begin{aligned} \nabla \cdot (\nabla \mathbf{u}) &= \frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u}) = 0 \\ \nabla \cdot (\nabla \mathbf{u}^T) &= \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j} = \Delta \mathbf{u} \end{aligned}$$

$$\begin{cases} -\mu\triangle\mathbf{u} + \nabla p & = \rho\mathbf{g} \\ \nabla \cdot \mathbf{u} & = 0 \end{cases}$$

The Biharmonic Equation

- for 2D, incompressible flow: $\mathbf{u} = (u, 0, w)$ and $\nabla \cdot \mathbf{u} = 0$
- there is a *streamfunction* ψ such that $\psi_z = u$, $-\psi_x = w$.
- take the curl of the Stokes eqn

$$\nabla \times \Big[-\mu \triangle \mathbf{u} + \nabla p = \rho \mathbf{g} \Big]$$

to get the *biharmonic equation*

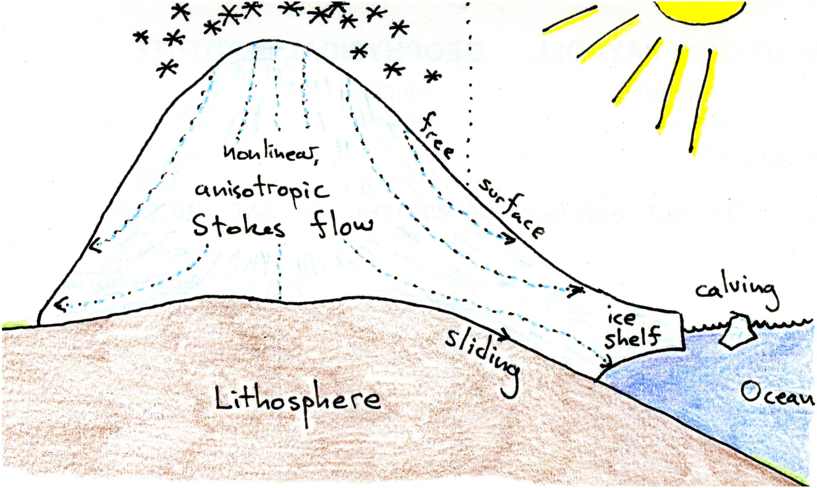
$$\psi_{xxxx} + 2\psi_{xxzz} + \psi_{zzzz} = 0$$

or

$$\triangle \triangle \psi = 0.$$

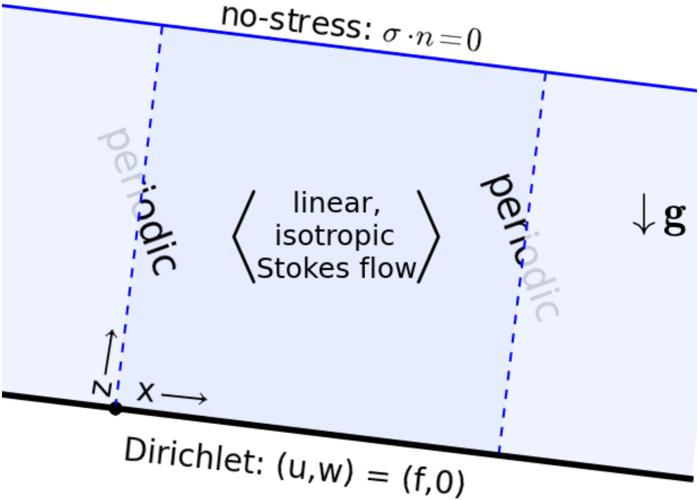
Quiz: give an example of a function solving the biharmonic eqn.

Ice: an awesome problem



...velocity, pressure, temperature, free surface all evolve

Slab-on-a-slope: a tractable problem



$$\mathbf{g} = (g_1, g_2) = \rho g \big(\sin(\alpha), -\cos(\alpha) \big)$$

...no evolution

Stokes bvp

find a velocity $\mathbf{u} = (u, w)$ and pressure p such that

$$
\begin{aligned}
-\nabla p + \mu \triangle \mathbf{u} &= -\mathbf{g} && \text{on } \Omega \\
\nabla \cdot \mathbf{u} &= 0 && \text{on } \Omega \\
\mathbf{u}(0, z) - \mathbf{u}(L, z) &= 0 && \text{for all } z \\
\mathbf{u}_x(0, z) - \mathbf{u}_x(L, z) &= 0 && \text{for all } z \\
u &= f && \text{on } \{z = 0\} \\
w &= 0 && \text{on } \{z = 0\} \\
w_x + u_z &= 0 && \text{on } \{z = H\} \\
2w_{zx} - (u_{xx} + u_{zz}) &= g_1/\mu && \text{on } \{z = H\}
\end{aligned}
$$

where
$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x)$.

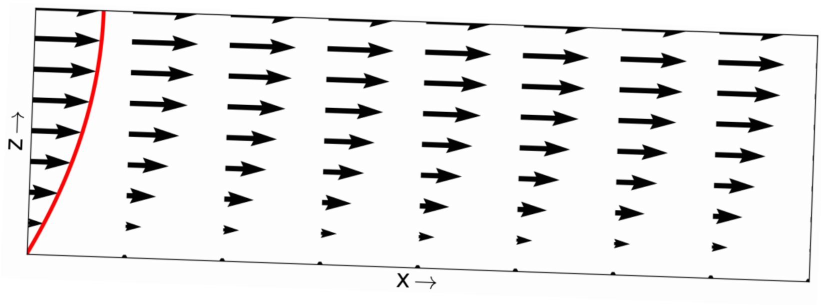
biharmonic bvp

find a streamfunction ψ such that

$$\begin{aligned}
\triangle\triangle\psi &= 0 && \text{on } \Omega \\
\psi_z(0,z) - \psi_z(L,z) &= 0 && \text{for all } z \\
\psi_{xz}(0,z) - \psi_{xz}(L,z) &= 0 && \text{for all } z \\
\psi_x(0,z) - \psi_x(L,z) &= 0 && \text{for all } z \\
\psi_{xx}(0,z) - \psi_{xx}(L,z) &= 0 && \text{for all } z \\
\psi(x,0) &= 0 && \text{for all } x \\
\psi_z(x,0) &= f && \text{for all } x \\
\psi_{zz}(x,H) - \psi_{xx}(x,H) &= 0 && \text{for all } x \\
3\psi_{xxz}(x,H) + \psi_{zzz}(x,H) &= -g_1/\mu && \text{for all } x.
\end{aligned} \tag{1}$$

biharmonic bvp, subproblem: $f = 0$

$$\psi(x,z) = \frac{g_1 H}{2\mu} z^2 - \frac{g_1}{6\mu} z^3 \quad \longrightarrow \quad \boxed{\begin{aligned} u(x,z) &= \frac{g_1 H}{\mu} z - \frac{g_1}{2\mu} z^2 \\ w(x,z) &= 0 \end{aligned}}$$



...this is Newtonian *laminar flow*, a well known solution.

biharmonic bvp, subproblem: $f \neq 0$

strategy:

- separate variables: $\psi(x, z) = X(x)Z(z)$
- periodicity: take $X(x) = \sin(\lambda x) + \cos(\lambda x)$ for $\lambda = \dfrac{2\pi n}{L}$
- the biharmonic eqn reduces to an ODE:

$$0 = \triangle^2(XZ) = X\left[\lambda^4 Z - 2\lambda^2 Z'' + Z^{(\text{iv})}\right]$$

- for $\lambda > 0$ this gives

$$Z(z) = a\sinh(\lambda z) + b\cosh(\lambda z) + cz\sinh(\lambda z) + dz\cosh(\lambda z)$$

- homogeneous bcs determine b, c, d in terms of a
- weighted sum gets the nonzero condition

Exact Solutions

Horizontal Component of Velocity:

$$u(x,z) = a_0 + \frac{g_1 H}{\mu} z - \frac{g_1}{2\mu} z^2 + \sum_{n=1}^{\infty} \frac{\lambda_n H^2 (a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x))}{\lambda_n^2 H^2 + \cosh^2(\lambda_n H)} Z_n'(z)$$

where

$$\begin{aligned} Z_n'(z) = {} & -\frac{1}{H} \cosh(\lambda_n H) \Big(\sinh(\lambda_n (z-H)) + \lambda_n z \cosh(\lambda_n (z-H)) \Big) \\ & + \frac{\cosh(\lambda_n H) - \lambda_n H \sinh(\lambda_n H)}{\lambda_n H^2} \cdot \Big(\cosh(\lambda_n (z-H)) \\ & + \lambda_n z \sinh(\lambda_n (z-H)) \Big) + \lambda_n \cosh(\lambda_n z). \end{aligned}$$

Exact Solutions

Vertical Component of Velocity:

$$w(x,z) = \sum_{n=1}^{\infty} \frac{\lambda_n^2 H^2}{\lambda_n^2 H^2 + \cosh^2(\lambda_n H)} (b_n \sin(\lambda_n x) - a_n \cos(\lambda_n x)) Z_n(z)$$

where

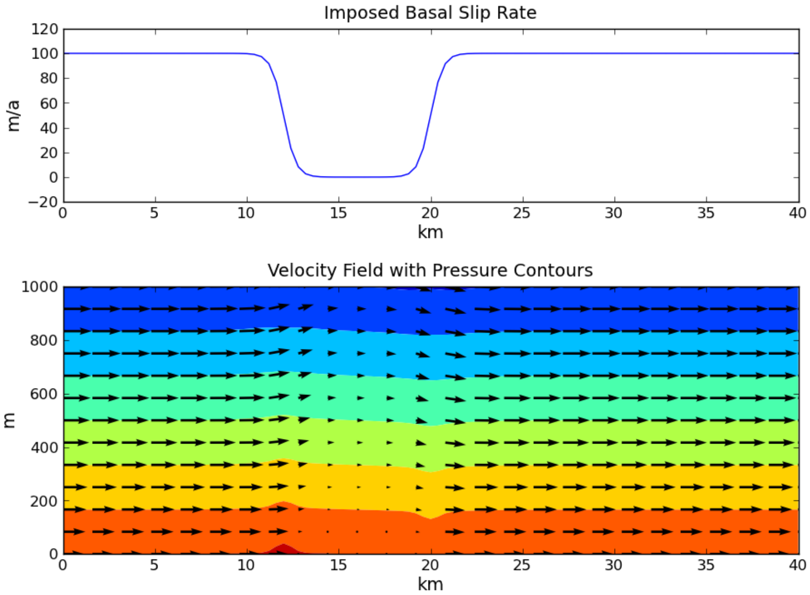
$$\begin{aligned} Z_n(z) = \sinh(\lambda_n z) - \frac{1}{H} \cosh(\lambda_n H) z \sinh(\lambda_n (z - H)) \\ + \left(\frac{\cosh(\lambda_n H)}{\lambda_n H^2} - \frac{\sinh(\lambda_n H)}{H} \right) z \cosh(\lambda_n (z - H)) \end{aligned}$$

Exact Solutions

Pressure:

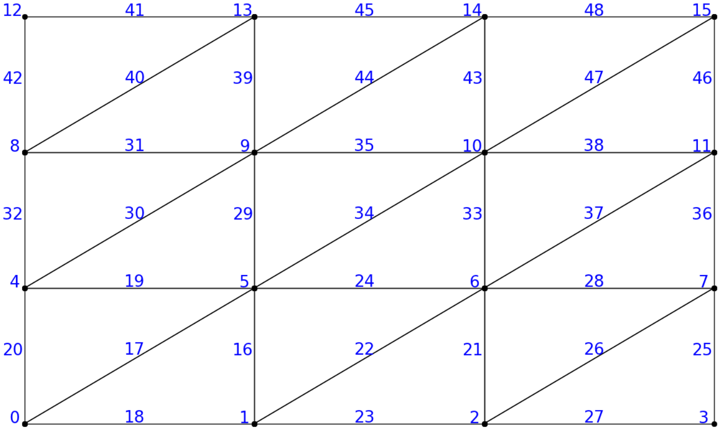
$$
\begin{aligned}
p(x,z) &= g_2 z - g_2 H \\
&\quad + 2\mu \sum_{n=1}^{\infty} \frac{\lambda_n^3 H (a_n \cos(\lambda_n x) - b_n \sin(\lambda_n x))}{\lambda_n^2 H^2 + \cosh^2(\lambda_n H)} \times \\
&\qquad \left[\sinh(\lambda_n z) - \frac{\cosh(\lambda_n H)}{\lambda_n H} \cosh(\lambda_n (z - H)) \right].
\end{aligned}
$$

... this is new.


Imposed Basal Slip Rate
m/a
km
Velocity Field with Pressure Contours
m
km

The finite element method

- numerical approximation of p and $\mathbf{u}$
- requires a mesh of the domain:



- leads to a system of linear equations $A\mathbf{x} = \mathbf{b}$

Variational Formulation: incompressibility

Incompressibility: $\nabla \cdot \mathbf{u} = 0$.

We seek a $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that for all $q \in L^2(\Omega)$, we have

$$\int_\Omega q \nabla \cdot \mathbf{u} = 0.$$

$\mathbf{u}$ is a *trial function*;
q is a *test function*.

Variational Formulation: Stokes

Put $\sigma = \tau - pI$ in the Stokes equation:

$$\mathbf{0} = \nabla \cdot \tau - \nabla p + \rho \mathbf{g}$$

Dot with $\mathbf{v} \in \mathbf{H}^1$ and integrate over Ω. *Integration by parts* gives

$$\int_\Omega \tau : \nabla \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} - \int_{\partial\Omega} \mathbf{n} \cdot \sigma \cdot \mathbf{v} = \rho \int_\Omega \mathbf{g} \cdot \mathbf{v}.$$

More manipulation gives

$$\frac{1}{2}\mu \int_\Omega \left(\nabla \mathbf{u}^T + \nabla \mathbf{u}\right) : \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right) - \int_\Omega p \nabla \cdot \mathbf{v} = \rho \int_\Omega \mathbf{g} \cdot \mathbf{v}.$$

Variational Formulation

Find $(\mathbf{u}, p) \in \mathbf{H}^1_E \times L^2$ such that for all $(\mathbf{v}, q) \in \mathbf{H}^1_{E_0} \times L^2$ we have

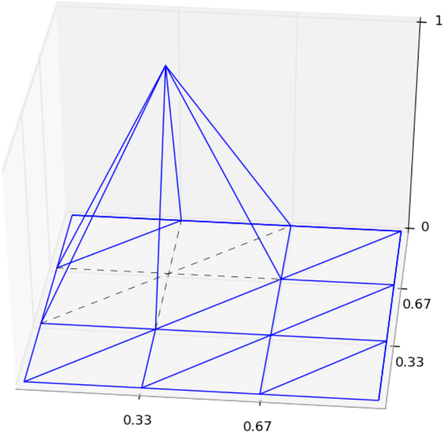
$$\frac{1}{2}\mu \int_\Omega \left(\nabla \mathbf{u}^T + \nabla \mathbf{u}\right) : \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right) - \int_\Omega p \nabla \cdot \mathbf{v} + \int_\Omega q \nabla \cdot \mathbf{u} = \rho \int_\Omega \mathbf{g} \cdot \mathbf{v}.$$

Still a continuous problem: $(\mathbf{u}, p)$ satisfy many conditions.

Make a discrete problem using finite-dimensional spaces.

Pressure Approximation space

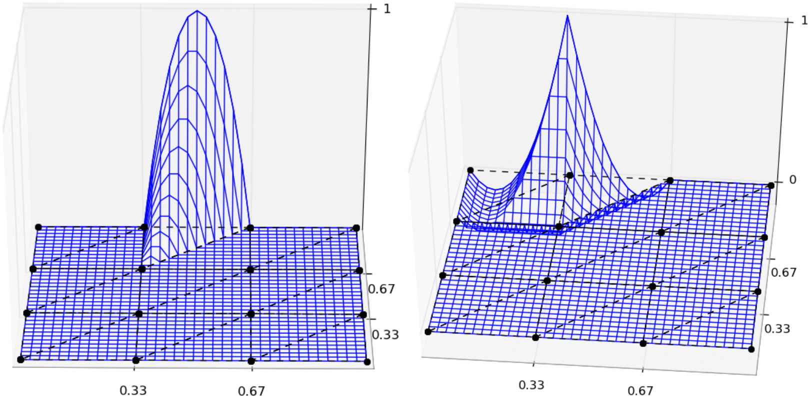
Continuous functions that are linear on each triangle:



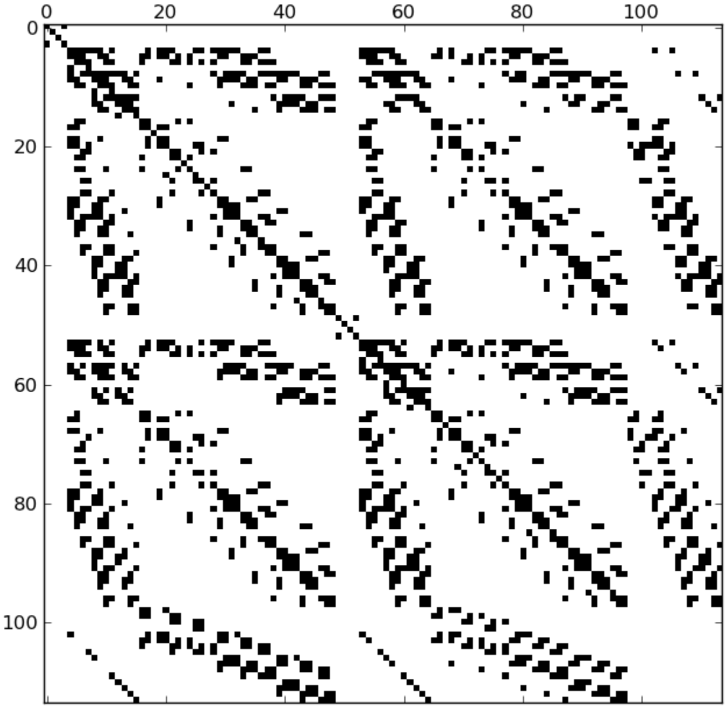
a 16-dimensional space with a convenient basis.

Velocity Approximation space

Continuous functions that are quadratic on each triangle:



a 49-dimensional space (per component) with a convenient basis.


0
20
40
60
80
100
0
20
40
60
80
100

Implementation I

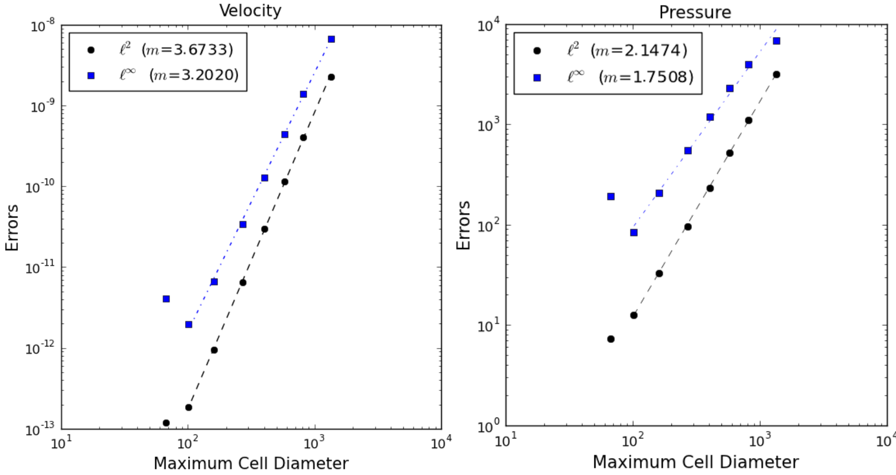
...based on [Jar08] but with an important difference

```python
from dolfin import *
#Set domain parameters and physical constants
Le,He = 4e3,  5e2  #length, height (m)
alpha = 1*pi/180   #slope angle (radians)
rho,g = 917,  9.81 #density (kg m-3), gravity (m sec-2)
mu    = 1e14       #viscosity (Pa sec)
G = Constant((sin(alpha)*g*rho,-cos(alpha)*g*rho))
#Define a mesh and some function spaces
mesh = Rectangle(0,0,Le,He,3,3)
V = VectorFunctionSpace(mesh, "CG", 2) #pw quadratic
Q =       FunctionSpace(mesh, "CG", 1) #pw linear
W = V * Q                              #product space
"""Define the Dirichlet condition at the base"""
def LowerBoundary(x, on_boundary):
    return x[1] < DOLFIN_EPS and on_boundary
SlipRate = Expression(("(3+1.7*sin(2*pi/%s*x[0]))\
                       /31557686.4"%Le,"0.0"))
bcD = DirichletBC(W.sub(0), SlipRate, LowerBoundary)
```

Implementation II

```
#Define the periodic condition on the lateral sides
class PeriodicBoundary_x(SubDomain):
    def inside(self, x, on_boundary):
        return x[0] == 0 and on_boundary
    def map(self, x, y):
        y[0] = x[0] - Le
        y[1] = x[1]
pbc_x = PeriodicBoundary_x()
bcP = PeriodicBC(W.sub(0), pbc_x)
"""Define the variational problem: a(u,v) = L(v)"""
(v_i, q_i) = TestFunctions(W)
(u_i, p_i) = TrialFunctions(W)
a = (0.5*mu*inner(grad(v_i)+grad(v_i).T, grad(u_i)\
    +grad(u_i).T) - div(v_i)*p_i + q_i*div(u_i) )*dx
L = inner(v_i, G)*dx
"""Matrix assembly and solution"""
U = Function(W)
solve(a==L,U,[bcD,bcP])
"""Split the mixed solution to recover u and p"""
(u, p) = U.split()
```

Convergence to Exact Solutions



Errors in FEM velocity and pressure plotted against maximum element diameter, together with convergence rates m.

References I

[Ach90] D. J. Acheson. *Elementary Fluid Dynamics*. Oxford University Press, New York, 1990.

[AG99] Cleve Ashcraft and Roger Grimes. SPOOLES: an object-oriented sparse matrix library. In *Proceedings of the Ninth SIAM Conference on Parallel Processing for Scientific Computing 1999 (San Antonio, TX)*, page 10, Philadelphia, PA, 1999. SIAM.

[Bat00] G. K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, UK, 2000.

[BR85] M. J. Balise and C. F. Raymond. Transfer of basal sliding variations to the surface of a linearly viscous glacier. *Journal of Glaciology*, 31(109), 1985.

[CP10] K. M. Cuffey and W. S. B. Paterson. *The Physics of Glaciers*. Academic Press, Amsterdam, Boston, 4th edition, 2010.

References II

[DH03] J. Donea and A. Huerta. *Finite Element Methods for Flow Problems*. Wiley, New York, 1st edition, 2003.

[DM05] L. Debnath and P. Mikusinski. *Introduction to Hilbert Spaces with Applications, Third Edition*. Academic Press, 3rd edition, 2005.

[Ern04] A. Ern. *Theory and Practice of Finite Elements*. Springer, Berlin, Heidelberg, 2004.

[ESW05] H. C. Elman, D. J. Silvester, and A. J. Wathen. *Finite Elements And Fast Iterative Solvers - With Applications in Incompressible Fluid Dynamics*. Oxford University Press, New York, 2005.

[GB09] R. Greve and H. Blatter. *Dynamics of Ice Sheets and Glaciers (Advances in Geophysical and Environmental Mechanics and Mathematics)*. Springer, 1st edition. edition, 8 2009.

References III

[Goo82] A. M. Goodbody. *Cartesian tensors - with applications to mechanics, fluid mechanics and elasticity*. E. Horwood, Chichester, 1982.

[Jar08] A. H. Jarosch. Icetools: A full stokes finite element model for glaciers. *Computers & Geosciences*, 34:1005–1014, 2008.

[Jóh92] Tómas Jóhannesson. *Landscape of Temperate Ice Caps*. PhD thesis, University of Washington, 1992.

[LJG+12] W. Leng, L. Ju, M. Gunzberger, S. Price, and T. Ringler. A parallel high-order accurate finite element nonlinear stokes ice sheet model and benchmark experiments. *Journal of Geophysical Research*, 117, 2012.

References IV

[LMW12] A. Logg, K. Mardal, and G. Wells, editors. *Automated Solution of Differential Equations by the Finite Element Method: The FEniCS Book (Lecture Notes in Computational Science and Engineering)*. Springer, Berlin, Heidelberg, 2 2012.

[Pir89] O. Pironneau. *Finite element methods for fluids*. Wiley, New York, 1989.

[QV94] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer, Berlin, Heidelberg, 1st edition, 1994.

[TS63] A. N. Tikhonov and A. A. Samarskii. *Equations of Mathematical Physics*. Pergamon Press, Oxford, 1963. Translated by A.R.M. Robson and P. Basu and edited by D. M. Brink.

[Wor09] G. Worster. *Understanding Fluid Flow*. Cambridge University Press, Cambridge, 1st edition, 2009.