Computing glacier geometry in nonlinear complementarity problem form

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(*funded by NASA Modeling, Analysis, and Prediction program*)

outline

1. NCPs and VIs, a superficial intro
2. glacier geometry-evolution models
3. every time-step is free-boundary problem
4. proposed approach: FVE discretization + Newton + continuation
5. partial success . . . and the essential difficulty

nonlinear complementarity problems (NCP)

- in finite dimensions, an NCP is to find $\mathbf{z} \in \mathbb{R}^n$ for which

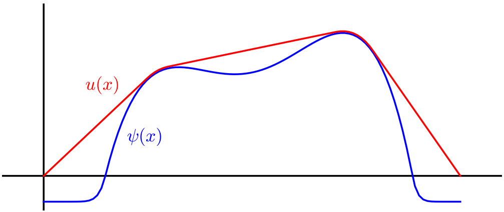
$$\mathbf{z} \geq 0, \quad \mathbf{F}(\mathbf{z}) \geq 0, \quad \mathbf{z}^\top \mathbf{F}(\mathbf{z}) = 0, \tag{1}$$

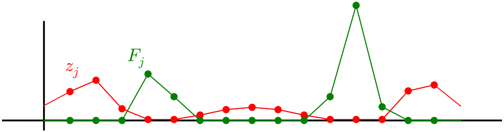
given a differentiable map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$

- example: given $\psi(x)$, the 1d obstacle problem is to find $u(x)$ so that $u(x) \geq \psi(x)$ and $-u''(x) = 0$ where $u > \psi$
- ...think about the gap ...
- discretized and in form (1):

$$z_j = u_j - \psi(x_j)$$

$$F_j(\mathbf{z}) = -\frac{z_{j+1} - 2z_j + z_{j-1}}{\Delta x^2} - \psi_j''$$





variational inequalities (VI)

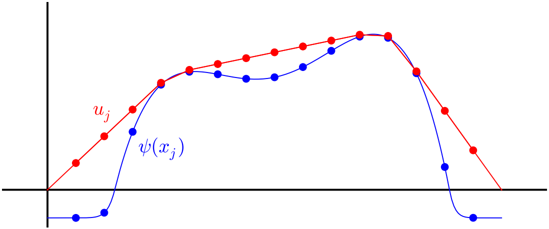
- in finite dimensions, a VI is to find $\mathbf{u} \in \mathcal{K}$, where $\mathcal{K} \subseteq \mathbb{R}^n$ is convex and closed, for which

$$\langle \mathbf{F}(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle \geq 0 \quad \forall \mathbf{v} \in \mathcal{K}, \tag{2}$$

  given a differentiable map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$

- obstacle problem:
  $\mathcal{K} = \{u_j \geq \psi(x_j)\}$ and

$$F_j(\mathbf{u}) = -\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$



NCP/VI generalities

- in finite dimensions when $\mathcal{K}$ is a cone (as in this talk):

$$\text{NCP} \iff \text{VI}$$

- both
  - generalize nonlinear eqns "$\mathbf{F}(\mathbf{z}) = 0$" to allow constraints on $\mathbf{z}$
  - are nonlinear, even if $\mathbf{F}$ is linear or affine
  - in practice: need iterative approach to solve
- constrained optimization $\implies$ VI $\iff$ NCP
  - i.e. find minimum of $\Phi[\mathbf{z}]$ from $\mathcal{K}$
  - symmetric Jacobian/Hessian in optimizations ($J = \mathbf{F}' = \Phi''$)
- *but*: NCP and VI arising in glacier problems are not optimizations

numerical support

libraries with scalable support for NCP and/or VI:

- PETSc SNES
  - does not assume optimization
  - used this in all results later in talk
- TAO
  - in PETSc release
  - separate code from SNES
- DUNE
  - used in 2011 . . . still maintained?

algorithms

two Newton line search NCP methods in PETSc SNES:[1]

- "reduced-space" = RS
    - active set $\mathcal{A} = \{i \,:\, z_i = 0 \text{ and } F_i(\mathbf{z}) > 0\}$
    - inactive set $\mathcal{I} = \{i \,:\, z_i > 0 \text{ or } F_i(\mathbf{z}) \le 0\}$
    - *algorithm*: compute Newton step $\mathbf{s}^k$ by

$$\left[J(\mathbf{z}^k)\right]_{\mathcal{I}^k,\mathcal{I}^k} \mathbf{s}_{\mathcal{I}^k} = -\mathbf{F}_{\mathcal{I}^k}(\mathbf{z}^k)$$

    then do projected line search onto $\{\mathbf{z} \ge 0\}$

- "semi-smooth" = SS
    - "NCP function":

$$\phi(a,b) = 0 \quad \iff \quad a \ge 0, b \ge 0, ab = 0$$

    - *algorithm*: compute Newton step $\mathbf{s}^k$ by

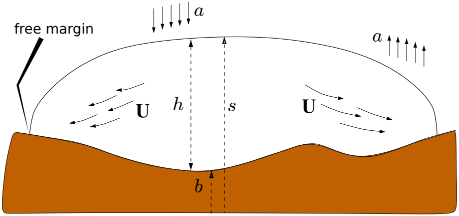
$$L^k \mathbf{s}^k = -\phi(\mathbf{z}^k, \mathbf{F}^k(\mathbf{z}^k))$$

    where L^k is element of $\partial_B \phi(\mathbf{z}^k, \mathbf{F}^k(\mathbf{z}^k))$; then do line search

[1] Benson & Munson (2006), and Barry Smith

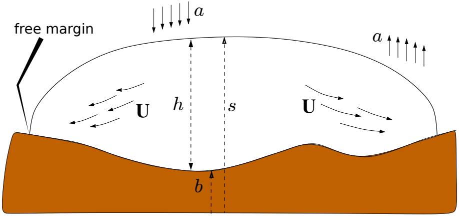
outline

glacier (and ice sheet) notation



- unknowns:
  - $h(t,x,y)$ ice thickness . . . also $s = h + b$ surface elevation
  - $\mathbf{U}(t,x,y,z) = \langle u, v, w\rangle$ ice velocity
- data:
  - $b(x,y)$ bed elevation
  - $a(t,x,y)$ surface mass balance
    - ⋆ accumulation/ablation function; $=$ precipitation $-$ melt
- ignored in this talk:
  - conservation of energy (temperature/enthalpy)
  - floating ice
  - solid-earth deformation

glacier (and ice sheet) notation



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- uncertain "data" from other models:
  - $b(x,y)$ bed elevation ? . . . improving for ice sheets
  - $a(t,x,y)$ surface mass balance ???
    - ★ accumulation/ablation function; = precipitation − melt
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solve coupled mass and momentum equations

- my goal: better ice sheet models
  - suitable for long/paleo ($\sim$ 100ka) and high res ($\sim$ 1 km)
  - without time-splitting
  - with explicit time-step restrictions
- here just two coupled conservations:
  - mass conservation

$$h_t + \nabla \cdot \mathbf{q} = a$$

    - ★ $\mathbf{q} = h \left< \bar{u}, \bar{v} \right>$ is vertically-integrated ice flux
    - ★ equivalent to “surface kinematical equation” (ice incompressible)
  - momentum conservation

$$\nabla \cdot \mathbf{U} = 0 \qquad \text{and} \qquad -\nabla \cdot \tau_{ij} + \nabla p - \rho\,\mathbf{g} = 0$$

    - ★ incompressible power-law Stokes ($D_{ij} = A\tau^{\nu-1}\tau_{ij}$ for $\nu = 3$)
    - ★ geometry (h & b) enters into boundary conditions

solve coupled mass and momentum equations

- my goal: better ice sheet models — vs PISM
  - suitable for long/paleo ($\sim$ 100ka) and high res ($\sim$ 1 km) — $\sim \checkmark$
  - without time-splitting — ☹
  - with explicit time-step restrictions — ☹
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many possible momentum equations

- incompressible power-law Stokes

$$\nabla \cdot \mathbf{U} = 0 \qquad \text{and} \qquad -\nabla \cdot \tau_{ij} + \nabla p - \rho\, \mathbf{g} = 0$$

- Blatter-Pattyn equations [η is effective viscosity]

$$-\nabla \cdot \left[\eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix}\right] + \rho g \nabla s = 0$$

- shallow shelf approximation (SSA)

$$-\nabla \cdot \left[\bar\eta h \begin{pmatrix} 4\bar u_x + 2\bar v_y & \bar u_y + \bar v_x \\ \bar u_y + \bar v_x & 2\bar u_x + 4\bar v_y \end{pmatrix}\right] - \tau_b + \rho g h \nabla s = 0$$

- non-sliding shallow ice approximation (SIA)

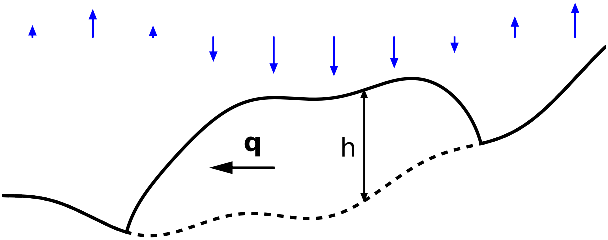
$$-\frac{\partial}{\partial z}\left[\eta \begin{pmatrix} u_z \\ v_z \end{pmatrix}\right] + \rho g \nabla s = 0 \qquad \to \qquad \langle \bar u, \bar v \rangle = -\Gamma h^{\nu+2} |\nabla s|^{\nu-1} \nabla s$$

- slow fluid momentum-conservation models all

  generate velocity $\mathbf{U} = \langle u, v, w \rangle$ from geometry h & b

- momentum equations are $\mathcal{M}(\mathbf{U}, h, b) = 0$

a fluid layer in a climate



- mass conservation equation on last slide applies to broader class:

  a fluid layer on a substrate, evolving in a climate

- mass conservation PDE:

$$h_t + \nabla \cdot \mathbf{q} = a \tag{$*$}$$

  - h is a thickness so $h \geq 0$
  - $(*)$ applies only where $h > 0$
  - signed source a is the "climate"

fluid layers in climates

glaciers

sea ice (& ice shelves)

tidewater marsh

tsunami inundation

fluid layers in climates

glaciers

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surface hydrology, subglacial hydrology, . . .

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semi-discretize in time

- semi-discretize coupled model (e.g. $h^\ell(x,y) \approx h(t^\ell,x,y)$)

$$\begin{aligned} h_t + \nabla\cdot\mathbf{q} &= a \\ \mathcal{M}(\mathbf{U},h,b) &= 0 \end{aligned} \quad \to \quad \begin{aligned} \tfrac{h^\ell - h^{\ell-1}}{\Delta t} + \nabla\cdot\mathbf{q}^\ell &= a^\ell \\ \mathcal{M}(\mathbf{U}^\ell,h^\ell,b) &= 0 \end{aligned}$$

- coupling also through $\mathbf{q} = \mathbf{q}(\mathbf{U},h)$
- details of flux $\mathbf{q}^\ell$ and source a^ℓ come from time-stepping scheme
    - backward-Euler shown
    - could use other θ-methods or BDFs
- need to weakly-pose single time-step mass conservation equation incorporating $h^\ell \geq 0$ constraint ... it generates the free boundary

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$$\begin{array}{ccc} h_t + \nabla\cdot\mathbf{q} = a & & \frac{h^\ell - h^{\ell-1}}{\Delta t} + \nabla\cdot\mathbf{q}^\ell = a^\ell \\ & \to & \\ \mathcal{M}(\mathbf{U},h,b) = 0 & & \mathcal{M}(\mathbf{U}^\ell,h^\ell,b) = 0 \end{array}$$

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mass conservation: VI form

- single time-step mass conservation equation

$$\frac{h^\ell - h^{\ell-1}}{\Delta t} + \nabla \cdot \mathbf{q}^\ell = a^\ell \qquad \text{(MC)}$$

- from now on: assume $\mathbf{q} = 0$ on any open set where $h = 0$
  - because it is a flowing *layer*
- first weak formulations of MC for glaciers were VIs
  - Calvo et al (2002): SIA 1d flat bed
  - Jouvet & Bueler (2012): SIA 2d general bed steady
- define $\mathcal{K} = \left\{ v \in W^{1,p}(\Omega) \,\middle|\, v \ge 0 \right\}$
- VI form of MC: find $h^\ell \in \mathcal{K}$

$$\int_\Omega h^\ell (v - h^\ell) - \Delta t\, \mathbf{q}^\ell \cdot \nabla (v - h^\ell) \ge \int_\Omega \left(h^{\ell-1} + \Delta t\, a^\ell \right) (v - h^\ell)$$

for all $v \in \mathcal{K}$

mass conservation: NCP form

- recall general NCP is

$$\mathbf{z} \geq 0, \quad \mathbf{F}(\mathbf{z}) \geq 0, \quad \mathbf{z}^\top \mathbf{F}(\mathbf{z}) = 0$$

- define

$$F(h) = h^\ell - h^{\ell-1} + \Delta t\, \nabla \cdot \mathbf{q}^\ell - \Delta t\, a^\ell$$

- NCP form of MC:

$$h^\ell \geq 0, \quad F(h^\ell) \geq 0, \quad h^\ell F(h^\ell) = 0$$

- setwise statements from the NCP:
  - where $h^\ell > 0$,

$$F(h^\ell) = 0 \qquad \iff \qquad \text{strong form MC}$$

    - ★ interior condition
  - where $h^\ell = 0$,

$$h^{\ell-1} + \Delta t\, a^\ell \leq 0$$

    - ★ says "surface mass balance is negative enough during time step to remove old thickness"

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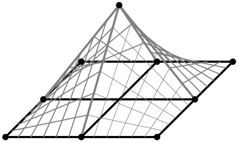
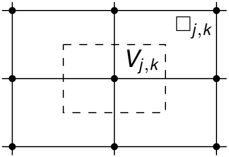
finite volume element (FVE) discretization

- from now on in this talk: steady case ($\Delta t = \infty$)
- for FVE, see Cai (1990) and Ewing, Lin, & Lin (2002)
- thickness $h(x, y)$ lives in Q^1 FEM space $\subset W^{1,\nu+1}(\Omega)$
  - structured grid for now; h bilinear on elements $\square_{j,k}$
- mass conservation $\iff$ control-volume integral on $V = V_{j,k}$:

$$\nabla \cdot \mathbf{q} = a \qquad \iff \qquad \int_{\partial V} \mathbf{q} \cdot \mathbf{n}\, ds \overset{*}{=} \int_V a\, dx\, dy$$

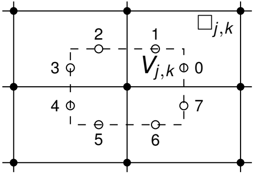
- *thus*: a finite element method where $*$ is the weak form
  - *or*: Petrov-Galerkin FEM with χ_V as test function
  - no symmetry in weak form . . . no loss





quadrature and upwinding

- FD schemes fit into above FVE framework
  - old FD scheme by Mahaffy (1976) fits . . . has weird quadrature
  - improved convergence comes from using quadrature points below:



- a bit of upwinding improves convergence on non-flat beds
  - . . . even though this is a fully-implicit approach
  - tested on bedrock-step exact solution (Jarosch et al 2013)
  - details out of scope here

restrict to SIA

- from now on: restrict to nonsliding SIA
- steady SIA mass conservation equation (SIA MC)

$$\nabla \cdot \mathbf{q} = a, \qquad \mathbf{q} = -\Gamma h^{\nu+2} |\nabla s|^{\nu-1} \nabla s$$

  - recall $s = h + b$
- main idea: subject to constraint $h \geq 0$, thus an NCP or VI

ad hoc continuation scheme

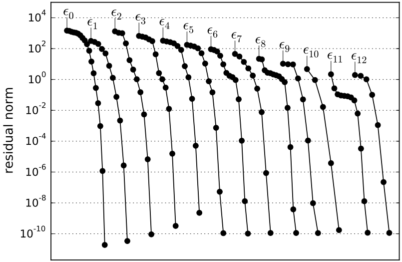
- for $0 \le \epsilon \le 1$, regularize $\mathbf{q}^{(\epsilon)}$ so that
  - $\epsilon_k = 10^{-k/3}$ for $k = 0, 1, \dots, 11$ and $\epsilon_{12} = 0$
  - $\mathbf{q}^{(\epsilon_0)}$ with $\epsilon_0 = 1$ gives classical obstacle problem

$$-\nabla \cdot (D_0 \nabla s) = a$$

  - $\mathbf{q}^{(\epsilon_{12})}$ with $\epsilon_{12} = 0$ gives SIA model

$$-\nabla \cdot (\Gamma h^{\nu+2} |\nabla s|^{\nu-1} \nabla s) = a$$

- in idealized cases, quadratic convergence at each level:



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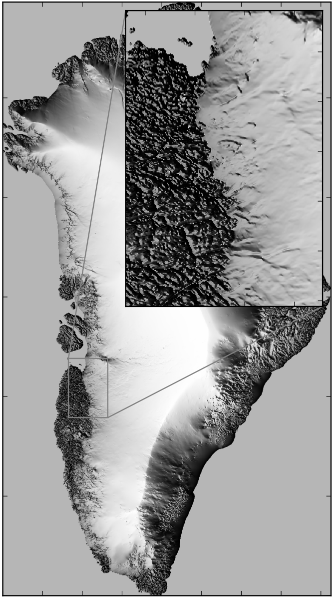
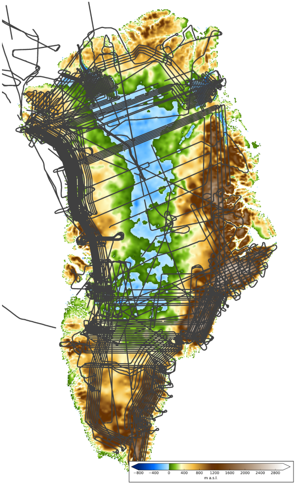
example: Greenland ice sheet

- *goal*: given steady surface mass balance $a(x, y)$ and bedrock elevation $b(x, y)$, predict the steady geometry $h(x, y)$ of the Greenland ice sheet
- *method*: solve steady SIA MC NCP
  - reduced-space Newton method
  - 900 m structured grid
  - Q^1 FEs in space
  - $N = 7 \times 10^6$ d.o.f.
- *result*: at right
  - see Bueler (2016), J. Glaciol.

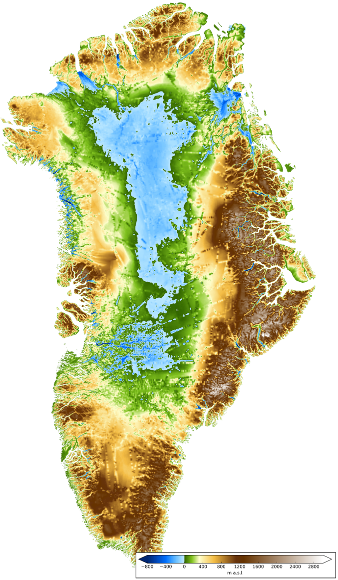


the essential difficulty: NASA’s darn airplanes

- actually: bedrock roughness



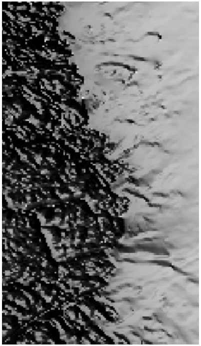
flight lines (OIB 2009-2014)

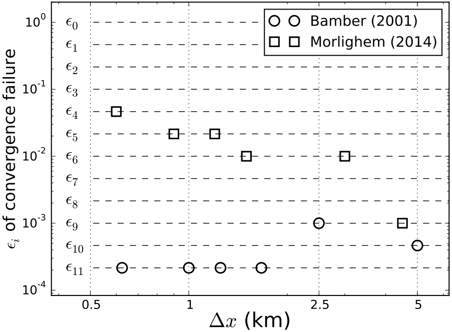


result: Morlighem (2014) bed map

convergence consequences

- improved bed observations $\Longrightarrow$ worse NCP solver convergence
  - old bed: Bamber (2001) on 5 km grid
  - new bed: Morlighem (2014) on 150 m grid
  - results shown for RS; SS is similar





rougher bed $\Longrightarrow$ poorer Newton-solver convergence

summary

- *problem*: fluid layer conservation model $h_t + \nabla \cdot \mathbf{q} = a$
  - subject to signed climate a
  - thickness h is nonnegative
  - coupled to momentum solver, for $\mathbf{U}$ in $\mathbf{q} = \mathbf{q}(\mathbf{U}, h)$
- *goals*:
  - long time steps, no first-order splitting errors
- *approach*:
  - take discrete-time, continuous-space seriously
  - pose single time-step weakly as NCP or VI
    - ★ incorporates constraint $h \geq 0$
    - ★ approach is largely flux-agnostic
  - solve by scalable constrained-Newton method (e.g. PETSc)
- *challenges*:
  - bed roughness makes convergence hard
  - every time step generates a near-fractal icy domain (e.g. continental ice sheet), via free-boundary problem, on which momentum solve must be accurate especially near the boundary