Fluid layers, climates, and weak formulations

Ed Bueler

Dept of Mathematics and Statistics, and Geophysical Institute
University of Alaska Fairbanks

(*funded by NASA Modeling, Analysis, and Prediction program*)

DMS Colloquium 5 April 2016

my motivation for this topic

- gradual realization, during sabbatical last year:

  there is a whole class of climate modeling problems which people are doing the wrong way

  - "people" = scientists/modelers who study cryosphere
  - special concern: choice of mathematical formulation

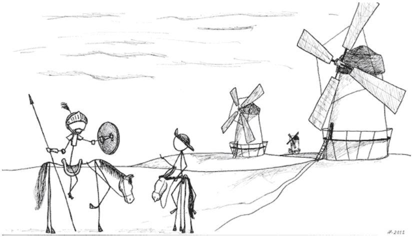
my motivation for this topic

- gradual realization, during sabbatical last year:

  there is a whole class of climate modeling problems which people are doing the wrong way

  - "people" = scientists/modelers who study cryosphere
  - special concern: choice of mathematical formulation

- i.e. watch me tilt at windmills



outline

glaciers and ice sheets

generalize to fluid layers and their climates

weak formulations

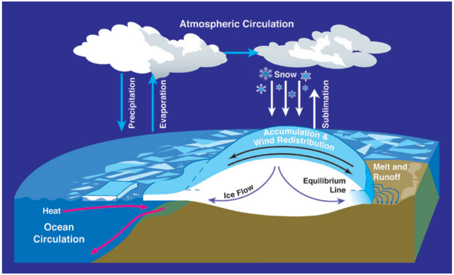
results

conservation: failures and fixes

conclusion

glaciers and ice sheets

- they move (flow and slide) under their own weight
  - speed 1 cm/yr – 10 km/yr
  - geometry determined in part by flow
- they accumulate snow, or melt, as sensitive function of global climate
  - Canadians should be grateful
- ice sheet = big glacier
- about 66 m (= 215 ft) of sea level rise equivalent
  - Antarctic ice sheet = 60 m
  - Greenland ice sheet = 6m
  - Alaska's glaciers $<$ 0.5 m



glacier model paradigms

- glacier problem (to most numerical modelers):

  given geometry of glacier, and stress boundary conditions, determine velocity of ice

  - a slow flow ($\mathrm{Re} \ll 1$) allows you to think this way

- glacier problem (to most actual glaciologists):

  given climate and topography, determine glaciated area and thicknesses of glaciers

glacier model paradigms

- glacier problem (to most numerical modelers):

  given geometry of glacier, and stress boundary conditions, determine velocity of ice

  - a slow flow ($\mathrm{Re} \ll 1$) allows you to think this way

- glacier problem (to most actual glaciologists):

  given climate and topography, determine glaciated area and thicknesses of glaciers

glacier model paradigms

- glacier problem (to most numerical modelers):

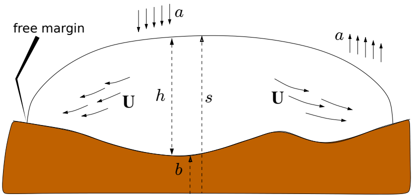
  given geometry of glacier, and stress boundary conditions, determine velocity of ice

  - a slow flow ($Re \ll 1$) allows you to think this way

- glacier problem (to most actual glaciologists):

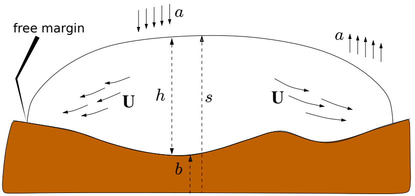
  given climate and topography, determine glaciated area and thicknesses of glaciers

glacier notation



- unknowns:
  - $h(t,x,y)$ ice thickness $\quad\ldots$ also $s = h + b$ surface elevation
  - $\mathbf{U}(t,x,y,z) = \langle u, v, w\rangle$ ice velocity
- data:
  - $b(x,y)$ bed elevation
  - $a(t,x,y,z)$ surface mass balance
    - a.k.a. accumulation/ablation function
    - $a =$ precipitation $-$ melt
- ignored in this talk:
  - conservation of energy (temperature/enthalpy)
  - floating ice

glacier notation



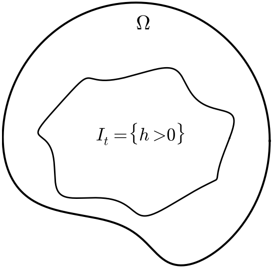
- unknowns:
  - $h(t,x,y)$ ice thickness ...also $s = h + b$ surface elevation
  - $\mathbf{U}(t,x,y,z) = \langle u, v, w \rangle$ ice velocity
- uncertain "data" from other models:
  - $b(x,y)$ bed elevation ...improving for ice sheets
  - $a(t,x,y,z)$ surface mass balance ...from GCM
    - a.k.a. accumulation/ablation function
    - $a =$ precipitation $-$ melt
- ignored in this talk:
  - conservation of energy (temperature/enthalpy)
  - floating ice

models solve coupled conservation equations

- *mass conservation*

$$h_t + \nabla \cdot \mathbf{q} = a$$

  - in ice-covered set $I_t = \{h > 0\} \subset \mathbb{R}^2$
    - changes in time
    - climate a and bed b defined on $\Omega \subset \mathbb{R}^2$ s.t. $I_t \subset \Omega$
  - $\mathbf{q}$ is vertically-integrated ice flux



- *momentum conservation*

$$\nabla \cdot \mathbf{U} = 0 \quad \text{and} \quad -\nabla \cdot \tau_{ij} + \nabla p - \rho\,\mathbf{g} = 0$$

  - in $E_t \subset \mathbb{R}^3$; changes in time
    - $I_t = \Pi_z E_t$
  - incompressible power-law Stokes
    - $D_{ij} = A\tau^2\tau_{ij}$
  - geometry (h & b) enters into b.c.s

many possible momentum equations

- incompressible Stokes

$$\nabla \cdot \mathbf{U} = 0 \qquad \text{and} \qquad -\nabla \cdot \tau_{ij} + \nabla p - \rho\,\mathbf{g} = 0$$

- Blatter-Pattyn equations [η is effective viscosity]

$$-\nabla \cdot \left[\eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix}\right] + \rho g \nabla s = 0$$

- shallow shelf approximation (SSA)

$$-\nabla \cdot \left[\bar\eta h \begin{pmatrix} 4\bar u_x + 2\bar v_y & \bar u_y + \bar v_x \\ \bar u_y + \bar v_x & 2\bar u_x + 4\bar v_y \end{pmatrix}\right] - \tau_b + \rho g h \nabla s = 0$$

- non-sliding shallow ice approximation (SIA)

$$-\frac{\partial}{\partial z}\left[\eta \begin{pmatrix} u_z \\ v_z \end{pmatrix}\right] + \rho g \nabla s = 0 \qquad \to \qquad \langle \bar u, \bar v\rangle = -\Gamma h^{\nu+2} |\nabla s|^{\nu-1} \nabla s$$

- slow-fluid momentum-conservation models always

  generate velocity $\mathbf{U} = \langle u, v, w\rangle$ from geometry h & b

- abstract momentum equations: $\quad \mathcal{M}(\mathbf{U}, h, b) = 0$

toward better models

- abstracted, mathematical model of glaciers:

$$\begin{aligned} h_t + \nabla \cdot \mathbf{q} &= a \qquad && \text{in } I_t = \{h > 0\} \subset \mathbb{R}^2 \\ \mathcal{M}(\mathbf{U}, h, b) &= 0 && \text{in } E_t \subset \mathbb{R}^3 \\ \mathbf{U} &= \langle u, v, w \rangle \\ \mathbf{q} &= \int_b^{h+b} \langle u, v \rangle \, dz \end{aligned}$$

- *my goal*: better numerical glacier models
  - effective for long runs ($\sim$ 100 ka) at high res ($\sim$ 1 km)
  - without first-order time-splitting errors
  - without explicit time-step restrictions

toward better models

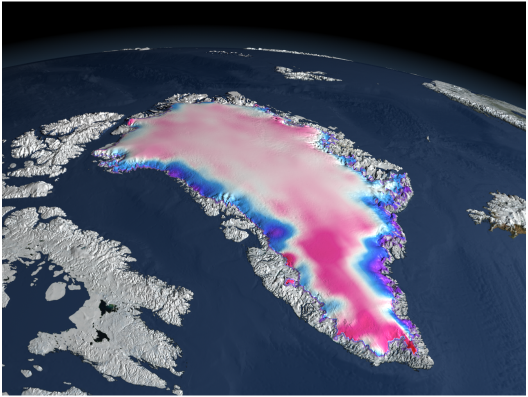
- abstracted, mathematical model of glaciers:

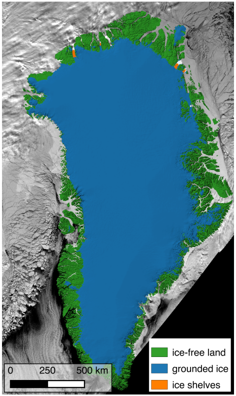
$$\begin{aligned} h_t + \nabla \cdot \mathbf{q} &= a \qquad \text{in } I_t = \{h > 0\} \subset \mathbb{R}^2 \\ \mathcal{M}(\mathbf{U}, h, b) &= 0 \qquad \text{in } E_t \subset \mathbb{R}^3 \\ \mathbf{U} &= \langle u, v, w \rangle \\ \mathbf{q} &= \int_b^{h+b} \langle u, v \rangle \, dz \end{aligned}$$

- *my goal*: better numerical glacier models
  - effective for long runs ($\sim$ 100 ka) at high res ($\sim$ 1 km) ☹
  - without first-order time-splitting errors ☹
  - without explicit time-step restrictions ☹
- vs PISM = Parallel Ice Sheet Model, `pism-docs.org`

practical numerical modeling

- some limitations in our big ice sheet model PISM:
  - explicit time-stepping
  - free boundary by truncation
- challenge: flowing ice (e.g. Greenland) is nearly-fractal





outline

glaciers and ice sheets

generalize to fluid layers and their climates

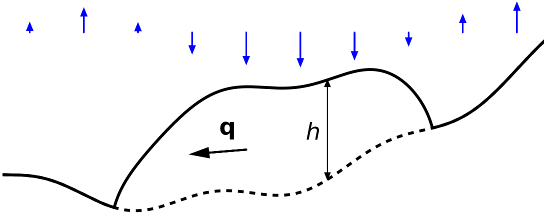
weak formulations

results

conservation: failures and fixes

conclusion

a fluid layer



- mass conservation eqn applies to broader class:

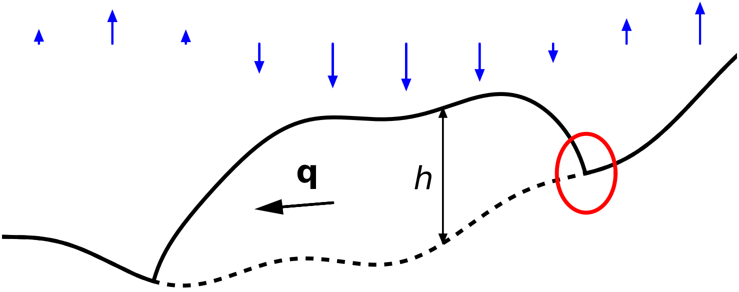
  a fluid layer on a substrate, evolving in a climate

- mass conservation PDE:

$$h_t + \nabla \cdot \mathbf{q} = a$$

  - h is a thickness so $h \geq 0$
  - signed source a is the "climate"; $a > 0$ shown downward
  - hidden, but important, part of the model: $\mathbf{q} = \mathbf{q}[h, b]$
  - PDE applies only where $h > 0$

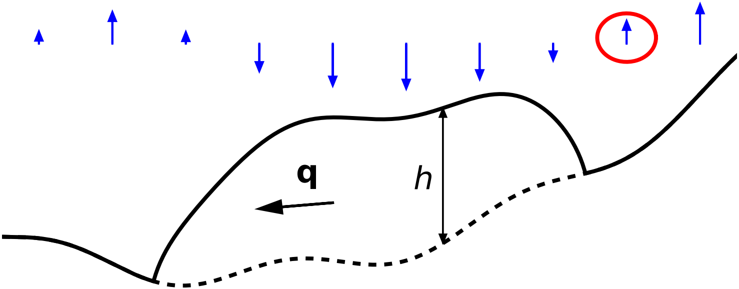
fluid layers: *the troubles*



$$h_t + \nabla \cdot \mathbf{q} = a$$

- $h = 0$ *and what else* at free boundary?
  - geometry at free boundary depends on both $\mathbf{q}$ and a
- $a < 0$ not “detected” by model where $h = 0$
  - how to do mass conservation accounting here?
- $a \approx 0$ sensitive-threshold behavior
  - nucleate new fluid model here if $a < 0$ switches to $a > 0$

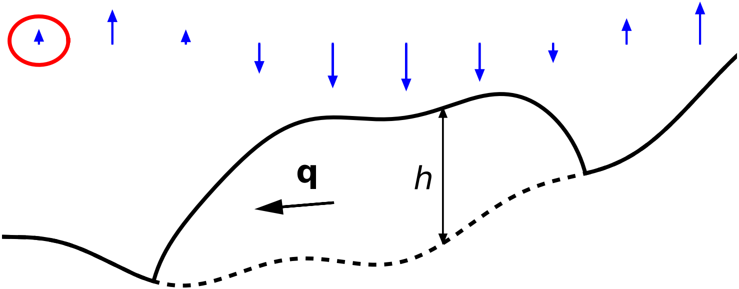
fluid layers: *the troubles*



$$h_t + \nabla \cdot \mathbf{q} = a$$

- $h = 0$ *and what else* at free boundary?
  - geometry at free boundary depends on both $\mathbf{q}$ and a
- $a < 0$ not “detected” by model where $h = 0$
  - how to do mass conservation accounting here?
- $a \approx 0$ sensitive-threshold behavior
  - nucleate new fluid model here if $a < 0$ switches to $a > 0$

fluid layers: *the troubles*



$$h_t + \nabla \cdot \mathbf{q} = a$$

- $h = 0$ *and what else* at free boundary?
  - geometry at free boundary depends on both $\mathbf{q}$ and a
- $a < 0$ not "detected" by model where $h = 0$
  - how to do mass conservation accounting here?
- $a \approx 0$ sensitive-threshold behavior
  - nucleate new fluid model here if $a < 0$ switches to $a > 0$

actual glacier free boundary

Martin Truffer, Taku Glacier SE Alaska, March 2016

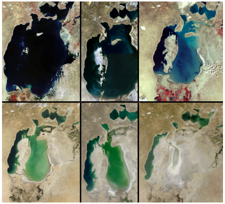
examples: fluid layers in climates

glaciers & ice sheets

sea ice

tidewater marsh

tsunami inundation



Aral Sea

time scales for major geometry changes

	lateral transport	*accumulation/ablation*
glacier/ice sheet	100 years	100 years
sea ice	1 week?	1 month?
tidewater marsh	1 hour	1 day?
tsunami	10 seconds	1 year
Aral Sea	?	1 year

- consider time-scales for major changes in geometry of the fluid layer from two sources:
  - motion of the fluid from forces applied at boundary (or body forces); *lateral transport*
  - *accumulation* (e.g. precipitation or aggregation) and *ablation* (melting, evaporation, sublimation)
- this talk: cases where these time-scales are comparable

time scales for major geometry changes

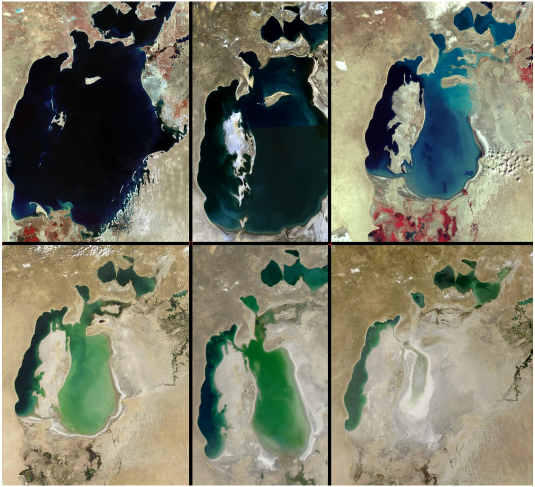
	lateral transport	*accumulation/ablation*
glacier/ice sheet	100 years	100 years
sea ice	1 week?	1 month?
tidewater marsh	1 hour	1 day?
~~tsunami~~	~~10 seconds~~	~~1 year~~
~~Aral Sea~~	~~?~~	~~1 year~~

- consider time-scales for major changes in geometry of the fluid layer from two sources:
  - motion of the fluid from forces applied at boundary (or body forces); *lateral transport*
  - *accumulation* (e.g. precipitation or aggregation) and *ablation* (melting, evaporation, sublimation)
- this talk: cases where these time-scales are comparable

case: tsunamis

- flux **q** dominates . . . climate *a* irrelevant

case: Aral Sea on Kazakhstan/Uzbekistan border



- top row: 1964, 1989, 1995?
- bottom row: 1999, 2002?, 2014
- climate a, and bdry fluxes, dominate . . . flux $\mathbf{q}$ irrelevant

fluid layers in climates: my actual examples

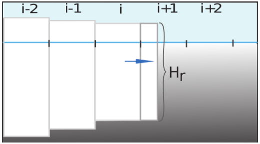
glaciers & ice sheets ✓

sea ice ✓

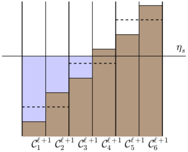
tidewater marsh ?

anyone numerically modeled this situation before?

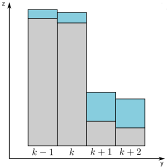
- yes, of course! generic results:
  - *ad hoc* schemes near the free boundary
  - only numerics; almost no continuum modeling



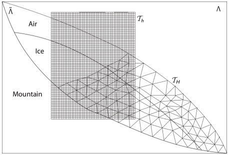
volume-of-fluid method at ice shelf fronts
(Albrecht et al, 2011)



tsunami run-up on shore
(LeVeque, George, Berger, 2011)



glacier ice on steep terrain
(Jarosch, Schoof, Anslow, 2013)



volume-of-fluid method at glacier surface
(Jouvet et al 2008)

outline

nonlinear complementarity problems (NCP)

- given differentiable $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, the NCP seeks $\mathbf{z} \in \mathbb{R}^n$ s.t.

$$\mathbf{z} \geq 0, \quad \mathbf{F}(\mathbf{z}) \geq 0, \quad \mathbf{z}^\top \mathbf{F}(\mathbf{z}) = 0$$

example NCP

- 1d obstacle problem: given $\psi(x)$, find $u(x)$ so that

$$u(x) \geq \psi(x)$$

  and

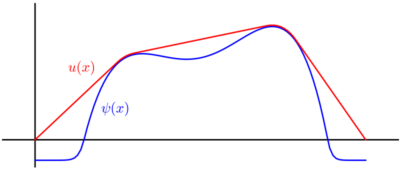
$$-u''(x) = 0 \quad \text{where } u > \psi$$

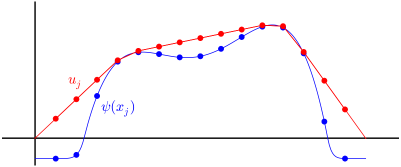
- discretize ...
- note $-u'' \geq 0$ ... and think about the gap ...
- in NCP form:

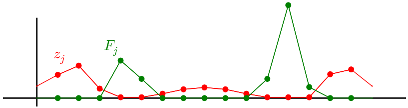
$$z_j = u_j - \psi_j \geq 0$$

$$F_j(\mathbf{z}) = -\frac{z_{j+1} - 2z_j + z_{j-1}}{\Delta x^2} - \psi_j'' \geq 0$$

$$\sum_j z_j F_j(\mathbf{z}) = 0$$







variational inequalities (VI)

- more general than NCP
- suppose $\mathcal{K} \subseteq \mathbb{R}^n$ is convex and closed
- given differentiable $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, the VI seeks $\mathbf{u} \in \mathcal{K}$ s.t.

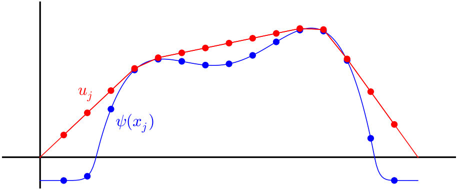
$$\langle \mathbf{F}(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle \geq 0 \qquad \forall \mathbf{v} \in \mathcal{K}$$

- also makes sense in ∞ dimensions

- 1d obstacle problem:

$$\mathcal{K} = \{u_j \geq \psi_j\}$$

$$F_j(\mathbf{u}) = -\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$



NCP/VI generalities

- suppose $\mathcal{K} \subseteq \mathbb{R}^n$ is a cone (always in this talk)
- then

$$\text{NCP} \iff \text{VI}$$

- both formulations
  - generalize nonlinear eqns $\mathbf{F}(\mathbf{x}) = 0$ to allow constraints
  - are nonlinear, even if $\mathbf{F}$ is linear or affine
  - in practice: need iterative approach to solve
- (convex optimization) $\implies$ VI $\iff$ NCP
  - i.e. find minimum of $\Phi[\mathbf{z}]$ from $\mathcal{K}$
  - symmetric Jacobian/Hessian ($\mathbf{F}' = \Phi''$)
  - *but*: NCP/VI arising in fluid layer problems are never optimizations (to my knowledge)

semi-discretize in time

- back to our fluid layer coupled model ...
- from now on: assume $\mathbf{q} = 0$ on any open set where $h = 0$
  - because it is a flowing *layer*
- semi-discretize:

$$\begin{aligned} h_t + \nabla \cdot \mathbf{q} &= a \\ \mathcal{M}(\mathbf{U}, h, b) &= 0 \end{aligned} \quad \to \quad \begin{aligned} \frac{h^\ell - h^{\ell-1}}{\Delta t} + \nabla \cdot \mathbf{q}^\ell &= a^\ell \\ \mathcal{M}(\mathbf{U}^\ell, h^\ell, b) &= 0 \end{aligned}$$

  - $h^\ell(x, y) \approx h(t^\ell, x, y)$
  - coupling through $\mathbf{q} = \mathbf{q}(\mathbf{U}, h)$
- this is still a continuum problem (in space)
- details of flux $\mathbf{q}^\ell$ and source a^ℓ come from scheme
  - backward-Euler shown
  - could use other θ-methods or BDFs

mass conservation: strong form . . . is inadequate

- single time-step mass conservation equation

$$\frac{h^\ell - h^{\ell-1}}{\Delta t} + \nabla \cdot \mathbf{q}^\ell = a^\ell \qquad \text{(MC)}$$

  - *strong form* of (MC)

- need to weakly-pose (MC), incorporating $h^\ell \geq 0$ constraint
  - seek $h^\ell(x, y)$ and location of the free boundary simultaneously

mass conservation: VI form

- first weak formulations of (MC) for glaciers were VIs
  - Calvo et al (2002): 1d glacier on flat bed
  - Jouvet & Bueler (2012): 2d glacier on general bed
- define $\mathcal{K} = \left\{ v \in W^{1,p}(\Omega) \,\middle|\, v \geq 0 \right\}$
- VI form of (MC): find $h^\ell \in \mathcal{K}$ s.t.

$$\int_\Omega h^\ell (v - h^\ell) - \Delta t\, \mathbf{q}^\ell \cdot \nabla (v - h^\ell) \geq \int_\Omega \left(h^{\ell-1} + \Delta t\, a^\ell \right) (v - h^\ell)$$

  for all $v \in \mathcal{K}$
  - derive from strong form (MC) by integration-by-parts plus thoughts about $h^\ell = 0$ areas

mass conservation: NCP form

- define

$$F(h) = h - h^{\ell-1} + \Delta t \, \nabla \cdot \mathbf{q} - \Delta t \, a$$

- NCP form of (MC): find h^ℓ s.t.

$$h^\ell \ge 0, \quad F(h^\ell) \ge 0, \quad h^\ell F(h^\ell) = 0$$

- setwise statements from the NCP:
  - where $h^\ell > 0$,

    $$F(h^\ell) = 0 \qquad \iff \qquad \text{strong form (MC)}$$

    - interior condition
  - where $h^\ell = 0$,

    $$h^{\ell-1} + \Delta t \, a^\ell \le 0$$

    - says "climate is sufficiently negative to remove old thickness" (during the time step)

numerical solution of the weak problem

for weak formulation (NCP or VI) of (MC):

- can be solved by a Newton method modified for constraint
- scalable & parallel implementations in C are in PETSc*
    - RS and SS methods for NCP (next slide)

*Portable Extensible Toolkit for Scientific computation, `www.mcs.anl.gov/petsc`

constrained-Newton algorithms

two Newton line search NCP methods:

- "reduced-space" = RS
  - inactive set $\mathcal{I} = \{i \,:\, z_i > 0 \text{ or } F_i(\mathbf{z}) \leq 0\}$
  - *algorithm*: compute Newton step $\mathbf{s}^k$ by

$$\left[J(\mathbf{z}^k)\right]_{\mathcal{I}^k,\mathcal{I}^k} \mathbf{s}_{\mathcal{I}^k} = -\mathbf{F}_{\mathcal{I}^k}(\mathbf{z}^k)$$

    then do projected ($\{\mathbf{z} \geq 0\}$) line search
- "semi-smooth" = SS
  - choose "NCP function":

$$\phi(a, b) = 0 \quad \iff \quad a \geq 0, b \geq 0, ab = 0$$

  - *algorithm*: compute Newton step $\mathbf{s}^k$ by

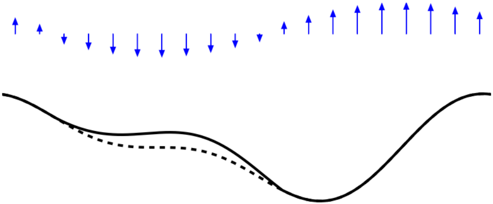
$$L^k \mathbf{s}^k = -\phi(\mathbf{z}^k, \mathbf{F}^k(\mathbf{z}^k))$$

    where L^k is element of $\partial_B \phi(\mathbf{z}^k, \mathbf{F}^k(\mathbf{z}^k))$; then do line search

1D time-stepping movies (and **q** flexibility)

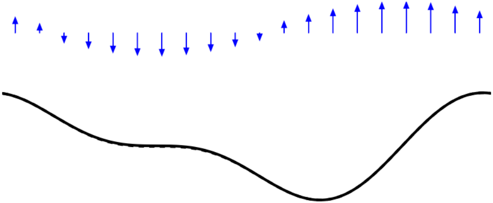
same:

- equation (MC)
- BEuler time-step
- climate a
- bed shape b
- constraint-respecting Newton scheme



top:

$\mathbf{q} = v_0 h$
hyperbolic advection with constant velocity



bottom:

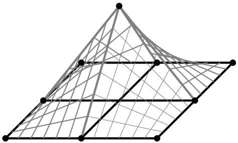
$\mathbf{q} = -\Gamma |h|^{n+2} \cdot |\nabla s|^2 \nabla s$
nonlinear degenerate diffusion

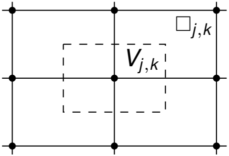
finite volume element (FVE) discretization

- thickness $h(x,y)$ lives in Q^1 FEM space $\subset W^{1,p}(\Omega)$
  - h is bilinear on elements $\square_{j,k}$
- (MC) discretized to control-volume integral on $V = V_{j,k}$
- in steady case ($\Delta t = \infty$) it looks like

$$\nabla \cdot \mathbf{q} = a \qquad \to \qquad \int_{\partial V} \mathbf{q} \cdot \mathbf{n}\, ds \overset{*}{=} \int_V a\, dx\, dy$$

  - $\Delta t = \infty$ is the hard case for numerics
- an FVE method is an FEM where $*$ is the weak form
  - *equiv*: Petrov-Galerkin FEM with 1_V as test function
  - no symmetry in weak form (…no loss for known cases)





restrict to shallow ice approximation (SIA) for glaciers

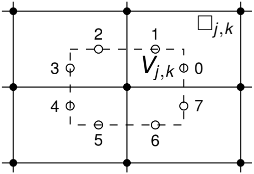
- from now on, restrict to nonsliding SIA:

$$\mathbf{q} = -\Gamma h^{\nu+2} |\nabla s|^{\nu-1} \nabla s$$

  - where $s = h + b$
  - justified by small-aspect-ratio argument

glacier numerics: quadrature and upwinding

- FD scheme for SIA fits into above FVE framework
  - i.e. Mahaffy (1976) ... but it has weird quadrature
  - improved convergence from quadrature points below:



  - these points good for any **q**, not just SIA
- a bit of upwinding improves convergence on non-flat beds
  - ... even though this is a fully-implicit approach
  - tested on bedrock-step exact solution (Jarosch et al 2013)
  - details out of scope here

Newton's method, regularized

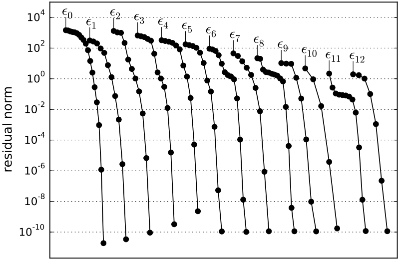
- for $0 \le \epsilon \le 1$, regularize $\mathbf{q}^{(\epsilon)}$
  - 12 levels: $\epsilon_0 = 1, \dots, \epsilon_{11} = 2 \times 10^{-4}, \epsilon_{12} = 0$
  - $\mathbf{q}^{(\epsilon_0)}$ with $\epsilon_0 = 1$ gives classical obstacle problem

$$-\nabla \cdot (D_0 \nabla s) = a$$

  - $\mathbf{q}^{(\epsilon_{12})}$ with $\epsilon_{12} = 0$ gives SIA model

$$-\nabla \cdot (\Gamma h^{\nu+2} |\nabla s|^{\nu-1} \nabla s) = a$$

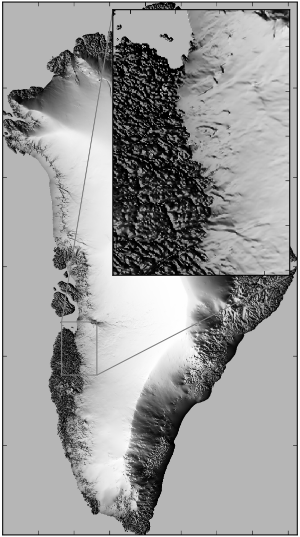
- at each level, use Newton's method
  - quadratic convergence for a dome test case:



outline

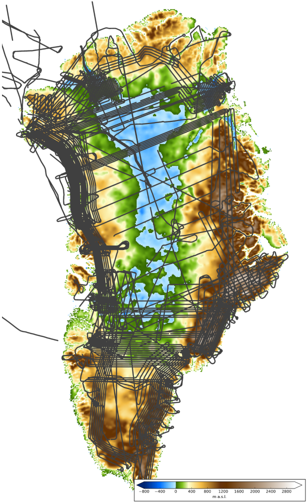
example: Greenland ice sheet

- *goal*: given steady surface mass balance $a(x, y)$, and bedrock elevation $b(x, y)$, predict the thickness $h(x, y)$ of the Greenland ice sheet
- *method* for steady SIA:
  - NCP form of (MC)
  - RS constrained-Newton
  - 900 m structured grid
  - Q^1 FEs in space
  - $N = 7 \times 10^6$ d.o.f.
- *result*: at right
  - see Bueler (2016), J. Glaciol.

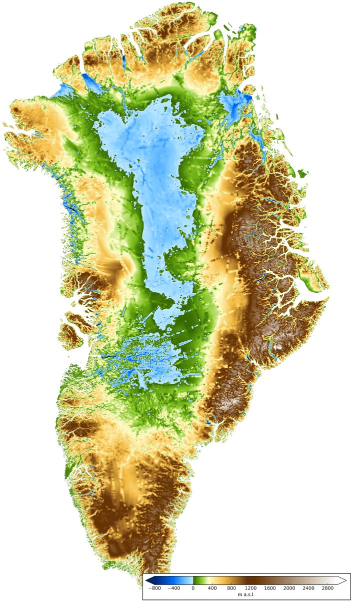


the essential difficulty: NASA’s darn airplanes

- actually: bedrock roughness



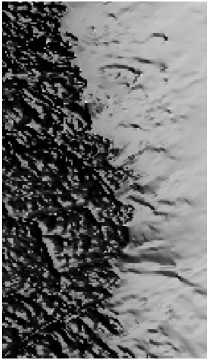
flight lines (OIB 2009-2014)



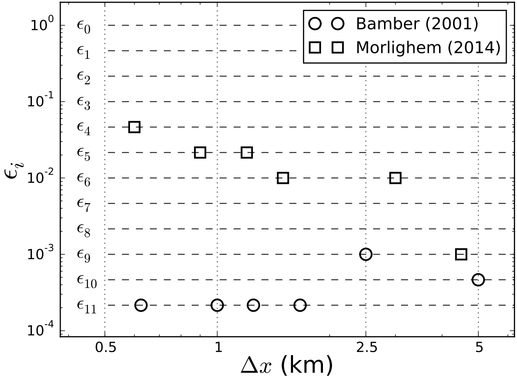
result: Morlighem (2014) bed map

convergence consequences

- improved observations $\implies$ worse solver convergence
  - old bed: Bamber (2001) on 5 km grid
  - new bed: Morlighem (2014) on 150 m grid
  - results shown for RS on NCP (MC); SS is similar



rougher bed $\implies$



poorer Newton convergence

outline

glaciers and ice sheets

generalize to fluid layers and their climates

weak formulations

results

conservation: failures and fixes

conclusion

my use of the word "climate"

- I have hijacked the word "climate" so as to define it to mean whatever I want
  - *typical mathematician behavior* ...

- my *Definition*: for a fluid layer model, with governing weak formulation as above (and NCP/VI), the *climate* is the function $a(t, x, y, z)$ defined on the whole domain Ω

- the climate is *only* a source term in the PDE "$h_t + \nabla \cdot \mathbf{q} = a$" in those locations where $h > 0$ (i.e. I_t)
- we should be able to report how much mass was transferred to/from the fluid layer by the climate

my use of the word "climate"

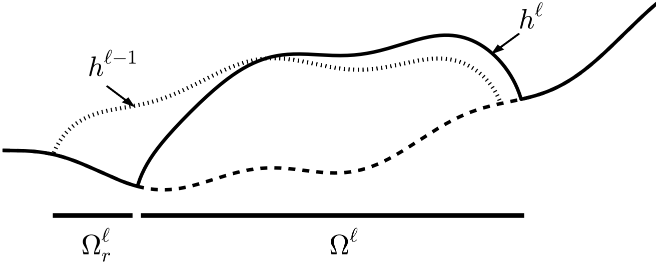
- I have hijacked the word "climate" so as to define it to mean whatever I want
  - *typical mathematician behavior* ...
  - you want a selfie? I'm not that kind of hijacker ...
- my *Definition*: for a fluid layer model, with governing weak formulation as above (and NCP/VI), the *climate* is the function $a(t, x, y, z)$ defined on the whole domain Ω
- the climate is *only* a source term in the PDE "$h_t + \nabla \cdot \mathbf{q} = a$" in those locations where $h > 0$ (i.e. I_t)
- we should be able to report how much mass was transferred to/from the fluid layer by the climate

conservation reporting: subsets

- back to time-stepping cases $\Delta t > 0$
- suppose h^ℓ solves the weak form (MC)
- define

$$\Omega^\ell = \left\{ h^\ell(x,y) > 0 \right\}$$

$$\Omega^\ell_r = \left\{ h^\ell(x,y) = 0 \text{ and } h^{\ell-1}(x,y) > 0 \right\} \quad \leftarrow \text{retreat set}$$



conservation reporting: time-series

- define:

$$M^\ell = \int_\Omega h^\ell(x,y)\,dx\,dy, \qquad \textit{the mass (volume) at time } t^\ell$$

- then

$$\boxed{\Delta t\,(-\nabla\cdot\mathbf{q}^\ell + a^\ell)}$$

$$\begin{aligned} M^\ell - M^{\ell-1} &= \int_{\Omega^\ell} \boxed{h^\ell - h^{\ell-1}}\,dx\,dy + \int_{\Omega_r^\ell} 0 - h^{\ell-1}\,dx\,dy \\ &= \Delta t\left(0 + \int_{\Omega^\ell} a^\ell\,dx\,dy\right) - \int_{\Omega_r^\ell} h^{\ell-1}\,dx\,dy \end{aligned}$$

  - this assumes $\mathbf{q} \to 0$ as $(x,y) \to \partial\Omega^\ell$
- new term:

$$R^\ell = \int_{\Omega_r^\ell} h^{\ell-1}\,dx\,dy \qquad \text{the } \textit{retreat loss during step } \ell$$

conservation reporting: the limitation

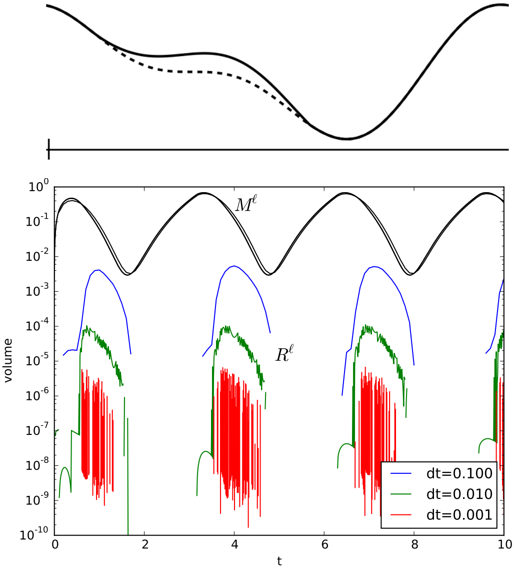
- the retreat loss R^ℓ is not balanced by the climate
  - retreat R^ℓ is *caused* by the climate, but we don't know a computable integral of a^ℓ to balance it
- we must track three time series:
  - mass at time t^ℓ: $M^\ell = \int_\Omega h^\ell(x,y)\,dx\,dy$
  - climate over current fluid-covered region:

$$C^\ell = \Delta t \int_{\Omega^\ell} a^\ell\,dx\,dy \qquad \approx \int_{t^{\ell-1}}^{t^\ell} \int_{I_t} a(t,x,y)\,dx\,dy\,dt$$

  - retreat loss during time step: $R^\ell = \int_{\Omega_r^\ell} h^{\ell-1}\,dx\,dy$
- and they balance:

$$M^\ell = M^{\ell-1} + C^\ell - R^\ell$$

conservation reporting: $R^\ell \to 0$ as $\Delta t \to 0$



outline

glaciers and ice sheets

generalize to fluid layers and their climates

weak formulations

results

conservation: failures and fixes

conclusion

summary

- *problem*: fluid layer MC model "$h_t + \nabla \cdot \mathbf{q} = a$"
  - layer thickness h
  - signed climate a
  - coupled to momentum solver: $\mathbf{q} = \mathbf{q}(\mathbf{U}, h)$
- *goals*:
  - long (implicit) time steps
  - rigorous conservation reporting
- *approach*:
  - pose single time-step problem weakly as NCP or VI
    - incorporates constraint $h \geq 0$
    - approach is **q**-agnostic
  - solve by scalable constrained-Newton method (e.g. PETSc)
- *challenges*:
  - bed roughness makes convergence hard
  - can this lead to global glaciation modeling?