Mathematical tools and obstacles in models of ice sheets

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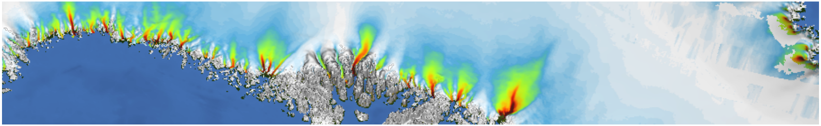


Denver University 15 May, 2015

Outline

I. an introduction to ice sheet flow for non-glaciologists

II. is the shallow ice problem well-posed and approximate-able?

III. can thin-layer free-boundary models exactly conserve?

ice in glaciers is a viscous fluid

- ... at least: glaciers are viscous flows at larger scales
- glaciers are free-surface flows (with no surface tension)
- *usage*: "ice sheets" = big, continent-scale glaciers

ice in glaciers is a viscous fluid

- primary variables: velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$
- also: ρ is density, $\mathbf{g}$ is gravity, ν is viscosity
- if the glacier fluid were "typical" like the ocean we would model with Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

$$\rho\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g} \qquad \textit{stress balance}$$

- but ice is not typical!
- e.g. not a worry in ice sheet flow models:
    - turbulence
    - coriolis force
    - convection

ice is a slow, shear-thinning viscous fluid

- glacier ice is
  1. "slow"[1]:

$$\rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) \approx 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix}\text{forces of inertia}\\ \text{are negligible}\end{pmatrix}$$

  2. non-Newtonian shear-thinning[2]:
     - viscosity ν is not constant
     - ν decreases as fluid is sheared

[1] $Fr \approx 10^{-15}$. Regarding coriolis: $Fr/Ro \approx 10^{-8}$.

[2] blood is also shear-thinning
corn starch in water is shear-thickening

the standard model

- slow flows are called "Stokes flows"
- notation:
  - τ_{ij} is deviatoric stress tensor
  - $\mathbf{D}u_{ij}$ is strain rate tensor
- the standard ice flow model is power-law Stokes:

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

$$0 = -\nabla p + \nabla \cdot \tau_{ij} + \rho \mathbf{g} \qquad \textit{slow stress balance}$$

$$\mathbf{D}u_{ij} = A \left|\tau_{ij}\right|^{n-1} \tau_{ij} \qquad \textit{Glen flow law}$$

- $1.8 < n < 4.0$?   when in doubt: $n = 3$
- $A > 0$ is "ice softness"
  - A varies strongly with temperature, but I'll ignore that here

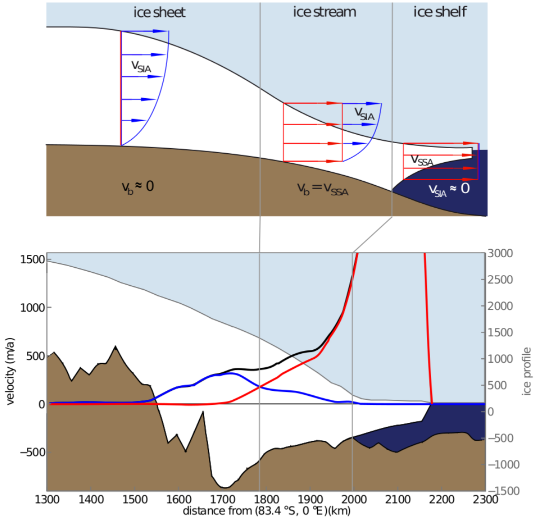
model design (because ice is a slow fluid)

- ice sheet modeling is a bit like atmosphere flow modeling or weather prediction, but . . .
- in a Stokes flow, the geometry, boundary stress, and viscosity determine velocity field and pressure, so . . .
- therefore a time-stepping ice sheet model recomputes the velocity field when needed, without requiring velocity from a previous time-step[3]

[3] to be a weatherman you've got to know which way the wind blows . . . but not to be a glaciologist

sheets versus streams versus shelves

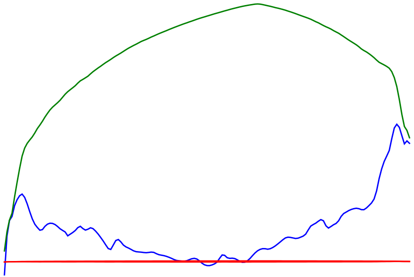
- non-sliding portions of ice sheets flow by shear deformation
- ice streams slide too
- "ice shelves" are floating thick ice ... all sliding
- ice shelves flow by extension not shear
- but: sliding ignored for today



[Lambert glacier and Amery ice shelf, Antarctic]

ice sheets are shallow

- cross-section of Greenland ice sheet at 71° N
  - green and blue: usual vertically-exaggerated version



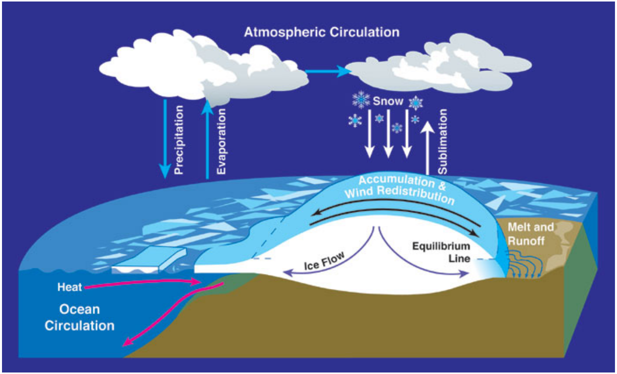
- in red: a view without this vertical exaggeration
- *thus*:
  - most simulations use shallow limits of Stokes
  - technical: high aspect-ratio elements bad in non-shallow solvers

first big picture: ice sheets store past climates

Greenland layers movie from NASA, January 2015

second big picture: ice sheets affect sea level

- *mass and energy inputs*: (1) snow adds, (2) sun heats, (3) ocean heats, (4) earth heats
- *mass outputs*: (1) surface meltwater, (2) basal meltwater, (3) ice discharge



summary so far

- ice sheets have five outstanding properties as viscous flows:
    1. free surface
    2. shallow
    3. slow
    4. shear-thinning
    5. sliding ("contact slip")
- remainder of my talk will address applied-mathematical questions for models which capture

part II properties 1, 2, 3, 4 but not 5

part III *any* time-dependent model which has properties 1 and 2

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shallow ice approximation (SIA)

- SIA = "lubrication" approximation of Stokes model
- good approximation when:
  - sliding is small or zero
  - bedrock slope is modest
- derive SIA equations by scaling Stokes using shallowness:
  - $[h]$ is a typical thickness scale
  - $[x]$ is a typical width scale
  - small parameter is $\epsilon = [h]/[x]$

SIA: velocity

- $b(x, y)$ is bedrock elevation (data)
- $s(x, y, t)$ is ice surface elevation (unknown)
- let $p = n + 1 > 2$
- assume: no sliding and isothermal
- horizontal ice velocity is given by:

$$\mathbf{U} = -\frac{p}{p+1}\Gamma\left[(s-b)^p - (s-z)^p\right]|\nabla s|^{p-2}\nabla s$$

where $\Gamma > 0$ combines gravity, ice density, ice softness

SIA plus mass conservation in steady state

- $a(x, y)$ is snowfall minus melt rate (data)
  - a.k.a. "surface mass balance"
- mass conservation in steady state:

$$\nabla \cdot \left(\int_b^s \mathbf{U}\, dz \right) = a$$

- shallow ice approximation + (steady) mass conservation:

$$-\nabla \cdot \left(\Gamma (s - b)^{p+1} |\nabla s|^{p-2} \nabla s \right) = a$$

  - this is "the SIA equation" for ice sheet geometry
  - like Poisson equation $-\nabla \cdot (D \nabla u) = f$
  - p-Laplace-ish ... but coefficient $(s - b)^{p+1} \to 0$ at margins
- equation above is simplest model for turning data of problem (b and a) into ice sheet geometry and velocity (s and U)

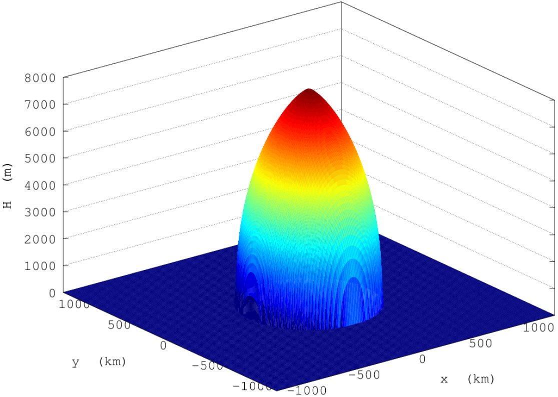
time-dependent SIA

- time-dependent SIA equation

$$\frac{\partial s}{\partial t} + \nabla \cdot \left(\Gamma (s-b)^{p+1} |\nabla s|^{p-2} \nabla s \right) = a$$

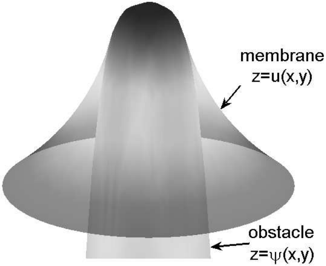
the Halfar (1983) similarity solution:

- an exact $b=0, a=0$ solution with $t \to 0^+$ limit a delta function
- compare Barenblatt solution of porous medium equation
- movie frames from $t=4$ months to $t=10^6$ years, equally-spaced in *exponential* time



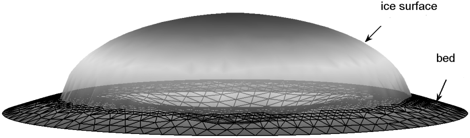
SIA: an analogy

- ice sheet surface $\sim$ membrane
- bedrock $\sim$ obstacle



- classical Laplacian obstacle problem:

$$\triangle u = 0 \quad \& \quad u \geq \psi$$



SIA equation is constrained

- $H = s - b$ is ice thickness
- thickness is a nonnegative quantity!
- steady state equation "strong form"

$$-\nabla \cdot \left(\Gamma H^{p+1} |\nabla s|^{p-2} \nabla s\right) = a$$

applies only where

$$s > b \qquad \iff \qquad H > 0$$

weak formulation by variational inequality

- define closed and convex admissible set

$$\mathcal{K} := \{\eta : \eta^{2p/(p-1)} \in W_0^{1,p}(\Omega) \text{ and } \eta \geq 0\}$$

  over larger domain $\Omega \subset \mathbb{R}^2$

- multiply strong form by $\eta - H$ where $\eta \in \mathcal{K}$, and integrate by parts thoughtfully

definition

$H \in \mathcal{K}$ solves the *steady shallow ice sheet problem* if

$$\int_\Omega \Gamma H^{p+1} |\nabla s|^{p-2} \nabla s \cdot \nabla(\eta - H) \geq \int_\Omega a(\eta - H)$$

for all $\eta \in \mathcal{K}$

ideas contained in weak formulation

- weak form

$$\int_\Omega \Gamma H^{p+1} |\nabla s|^{p-2} \nabla s \cdot \nabla(\eta - H) \geq \int_\Omega a(\eta - H)$$

  implies strong form

$$-\nabla \cdot \left(\Gamma H^{p+1} |\nabla s|^{p-2} \nabla s\right) = a$$

  where $H > 0$
- weak form also implies: if $H = 0$ then $a \leq 0$
- if $b = 0$ then the weak form is equivalent to a constrained minimization problem
  - for general bed b this is not so

on well-posedness (Jouvet-Bueler 2012)

theorem A.

if $b = 0$ then the variational inequality is equivalent to

$$\min\ J[u] = \int_\Omega \frac{\Gamma}{p} |\nabla u|^p - au$$

over $\eta = u^{(p-1)/(2p)} \in \mathcal{K}$, and this problem is well-posed (existence, uniqueness, stability w.r.t. data a)

theorem B.

in general case ($b \neq 0$) there exists a solution

- proof of B by a fixed-point theorem, of course
- we really don't know about uniqueness (not just technical)

an interesting quality of this variational inequality

- every glaciologist believes this about steady climates:

  if $a > 0$ on R then $H > 0$ on R

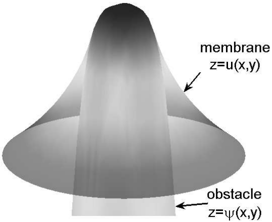
  that is:

  if it snows more than it melts then you get a glacier

- uniformly-elliptic variational inequalities, e.g. the classical obstacle problem,

  $$\int_\Omega \nabla u \cdot \nabla(v - u) \ge \int_\Omega f(v - u),$$

  for all $v \ge \psi$, do *not* have the analogous property



another weak form

- the variational inequality for shallow ice has abstract form

$$-\langle \mathbf{q}(H, \nabla H), \nabla(\eta - H)\rangle \geq \langle a, \eta - H\rangle \qquad \forall \eta \in \mathcal{K}$$

where $\mathbf{q} = \int_b^s \mathbf{U}\, dz = -\Gamma H^{p+1}|\nabla s|^{p-2}\nabla s$ is the “flux”

- a standard localization argument gives

$$F \geq 0 \quad \text{and} \quad H\, F = 0$$

where $F = \nabla \cdot \mathbf{q} - a$

finite-dimensional

- under (e.g.) finite element or volume discretization the shallow ice problem can be written:

$$H \in \mathbb{R}^m \qquad \text{and} \qquad F(H) \in \mathbb{R}^m$$
$$H \geq 0 \qquad \text{and} \qquad F(H) \geq 0$$
$$H\, F(H) = 0$$

- in unconstrained case we would just solve "$F(H) = 0$", so we call F the "residual"

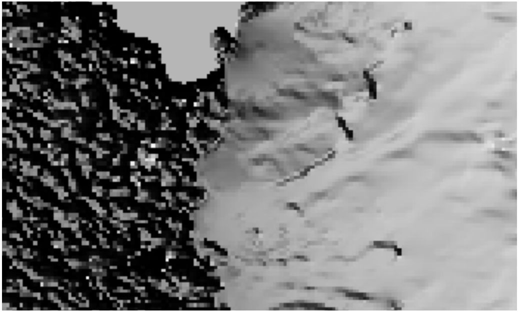
definition.

for $f : \mathbb{R}^m \to \mathbb{R}^m$, we say $x^* \in \mathbb{R}^m$ solves the *nonlinear complementarity problem* (NCP) if

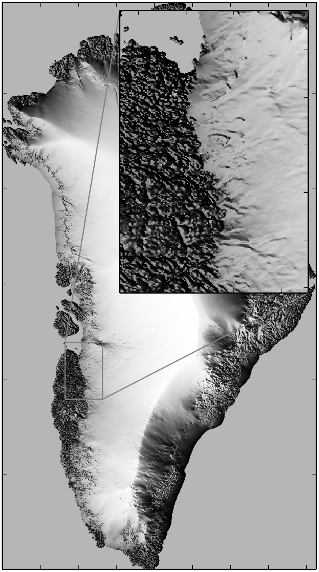
$$x^* \geq 0 \quad \text{and} \quad f(x^*) \geq 0 \quad \text{and} \quad x^*\, f(x^*) = 0$$

- scalable Newton solvers are available for NCPs
    - from PETSc

example computation: steady Greenland ice sheet

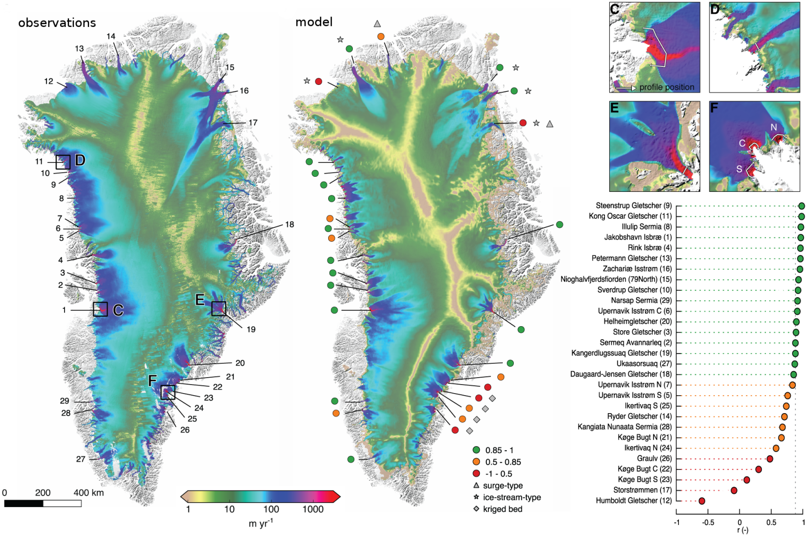


- data is $b(x, y)$ and $a(x, y)$
- 900 m grid on 1500×2600 km domain
- NCP solved on 192 processors
  - reduced-space Newton solver
  - double continuation by
    - diffusivity regularization
    - bed smoothing



Bueler (submitted)

Greenland velocity from a better model



Aschwanden et al (submitted); model is the Parallel Ice Sheet Model (`pism-docs.org`)

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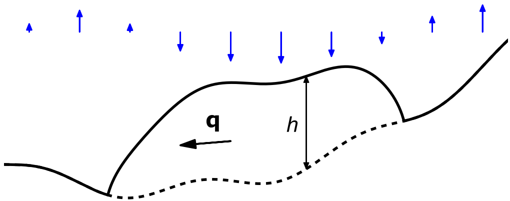
thin layers in a "climate"



- abstract problem, of which ice sheets are one case:

$$h_t + \nabla \cdot \mathbf{q} \overset{*}{=} f$$

on $\Omega \subset \mathbb{R}^2$, where $\mathbf{q} = \mathbf{q}(\nabla h, h, x, t)$ and $f = f(h, x, t)$

- h is layer thickness, $\mathbf{q}$ is flux, and f is source or "climate"
- PDE $*$ can be parabolic or hyperbolic . . . but it only applies where $h > 0$
- "$h \geq 0$" needs to be a constraint
  - generates free boundary somewhere in region where $f < 0$

examples

glaciers

ice shelves & sea ice

tidewater marsh

tsunami inundation (?)

and sea-level rise, surface hydrology, subglacial hydrology, . . .

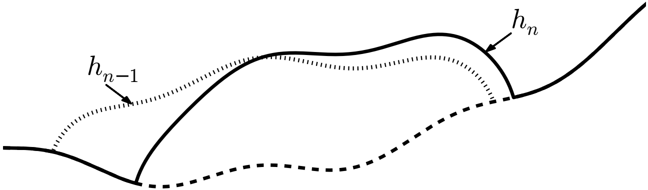
discrete time

- numerical models for this problem must discretize time:

$$h_t + \nabla \cdot \mathbf{q} = f \qquad \longrightarrow \qquad \frac{h_n - h_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

  - in minimal sense: semi-discretize in time
  - backward Euler, trapezoid, RK, ...
  - but remember constraint: $h_n \geq 0$

- cartoon:



a free-boundary problem for each time step

- strong form for time step:

$$\frac{h_n - h_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n, \qquad h_n \geq 0$$

- let $\mathcal{X}$ be a Sobolev space for which the unconstrained "Poisson" problem on Ω is well-posed:

$$\nabla \cdot \mathbf{Q}_n(\nabla v, v, x) = g$$

- let $\mathcal{K} = \{v \in \mathcal{X} \,|\, v \geq 0\}$
- weak form for time step: find $h_n \in \mathcal{K}$ so that

$$\langle A_n(h_n), v - h_n \rangle \geq 0 \quad \text{for all } v \in \mathcal{K}$$

where $\langle A_n(v), \phi \rangle := \int_\Omega (v - \Delta t\, F_n - h_{n-1})\, \phi - \Delta t\, \mathbf{Q}_n \cdot \nabla \phi$

questions

for the weak form (variational inequality)

$$\langle A_n(h_n), v - h_n \rangle \geq 0 \qquad \text{for all } v \in \mathcal{K}$$

we have these questions:

- *well-posedness*: are implicit time-steps well-posed?
  - explicit time steps are well-posed
    - just compute h_n and truncate to impose $h_n \geq 0$
    - but has severe stability time-step restrictions!
- *exact conservation*: can we exactly-balance the change in fluid mass over time step by computable integrals of f or F_n?
  - $M_n = \rho \int_\Omega h_n(x)\, dx$ is the fluid mass at time t_n

well-posedness for each time step

- the answer to the first question depends on the form of the flux $\mathbf{q}$ ... but it is “usually” yes

theorem (Bueler, in prep).

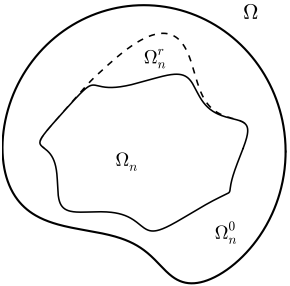
for a variety of flux forms $\mathbf{q}(\nabla h, h, x, t)$, a backward Euler time-step is well-posed

- p-Laplacian: $\mathbf{q} = -k|\nabla h|^{p-2}\nabla h$
- porous medium and doubly-nonlinear: $\mathbf{q} = -kh^r|\nabla h|^{p-2}\nabla h$
  - includes SIA in flat bed-rock case
- advecting layer: $\mathbf{q} = \mathbf{X}h$ where $\mathbf{X}$ is a fixed velocity field
- certain nonlocal, but linear, forms

for conservation: solution-dependent subsets of Ω

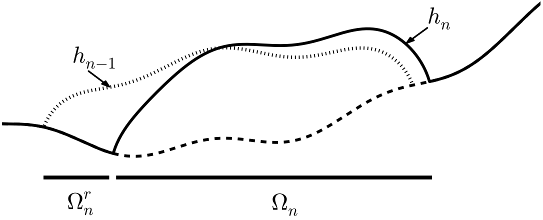
for the second question we define sets:

$$\Omega_n = \{h_n(x) > 0\}$$
$$\Omega_n^r = \{h_n(x) = 0 \text{ and } h_{n-1}(x) > 0\}$$
$$\Omega_n^0 = \{h_n(x) = 0 \text{ and } h_{n-1}(x) = 0\}$$



so that

$$\Omega = \Omega_n \cup \Omega_n^r \cup \Omega_n^0$$



the discrete-time conservation issue

- calculation:

$$\boxed{\Delta t \left(-\nabla \cdot \mathbf{Q}_n + F_n\right)}$$

$$M_n - M_{n-1} = \int_{\Omega_n} \boxed{h_n - h_{n-1}}\, dx + \int_{\Omega_n^r} 0 - h_{n-1}\, dx$$

$$= \Delta t \left(0 + \int_{\Omega_n} F_n\, dx\right) - \int_{\Omega_n^r} h_{n-1}\, dx$$

- define new term:

$$R_n = \int_{\Omega_n^r} h_{n-1}\, dx$$

is the retreat loss during time step n

bad news for exact discrete conservation

claim (Bueler, in prep).

- if the thin layer has a free boundary then you cannot exactly conserve, in the sense that there is no computable integral I_n of the source term f, over the time step, so that

$$M_n - M_{n-1} = I_n$$

- restore balance by adding the retreat loss *a posteriori*:

$$M_n - M_{n-1} = C_n - R_n$$

where $C_n = \Delta t \int_{\Omega_n} F_n \, dx$ is a computable integral of the source term and $R_n = \int_{\Omega_n^r} h_{n-1} \, dx$ is computable from h_n

- I wonder how to make this into an honest theorem . . .

practical conclusions for ice sheet modeling

- is steady shallow ice problem
  - well-posed? probably
  - approximate-able? yes
- can thin-layer free-boundary problems
  - be solved by implicit time-stepping? yes
  - be exactly discrete conserving? no
    - but error can be quantified