Weak and shallow: New thinking about simulations of ice sheet flows

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weak, shallow, and fairly new

- C. Schoof (**2006**) *A variational approach to ice stream flow*, J. Fluid Mech. 556, 227–251
- E. Bueler, J. Brown (**2009**) *Shallow shelf approximation as a "sliding law" in a thermodynamically coupled ice sheet model*, J. Geophys. Res. 114, F03008
- G. Jouvet, E. Bueler (**2012**) *Steady, shallow ice sheets as obstacle problems: well-posedness and finite element approximation*, SIAM J. Appl. Math. 72 (4), 1292–1314
- G. Jouvet, E. Bueler, C. Gräser, R. Kornhuber (**to appear**) *A nonsmooth Newton multigrid method for a hybrid, shallow model of marine ice sheets*, Proc. 8th ICSCA, AMS Cont. Math.

G. Jouvet = Guillaume Jouvet, Free University of Berlin

Outline

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ice sheet flow: an introduction for non-glaciologists

shallow ice approximation for grounded ice sheets

a model for ice streaming

a model for marine ice sheet evolution

ice in glaciers is a viscous fluid

- . . . at least: glaciers are viscous flows at larger scales
- *usage*: "ice sheets" are big, shallow glaciers

ice in glaciers is a viscous fluid

- primary variables: velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$
- also: ρ is density, $\mathbf{g}$ is gravity, ν is viscosity
- if the glacier fluid were "typical" like the ocean we would model with Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

$$\rho\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g} \qquad \textit{stress balance}$$

- but ice is not typical!
- e.g. not addressed in ice sheet flow models:
  - turbulence
  - convection
  - coriolis force
  - density-driven flow

ice is a slow, shear-thinning viscous fluid

- our glacier fluid is
  1. "slow"[1]:

$$\rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) \approx 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix}\text{forces of inertia}\\ \text{are negligible}\end{pmatrix}$$

  2. non-Newtonian (shear-thinning):

viscosity ν is not constant

[1] $Fr \approx 10^{-15}$. Regarding coriolis: $Fr/Ro \approx 10^{-8}$.

ice is a slow, shear-thinning viscous fluid

- notation:
  - τ_{ij} is deviatoric stress tensor
  - $\mathbf{D}u_{ij}$ is strain rate tensor
- the standard ice flow model is Glen-law Stokes:

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

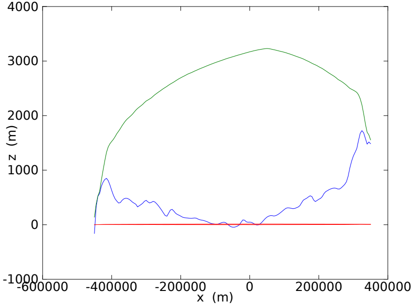
$$0 = -\nabla p + \nabla \cdot \tau_{ij} + \rho \mathbf{g} \qquad \textit{slow stress balance}$$

$$\mathbf{D}u_{ij} = A\,|\tau_{ij}|^{n-1}\,\tau_{ij} \qquad \textit{Glen flow law}$$

- $1.8 < n < 4.0$? when in doubt: $n = 3$
- $A > 0$ is “ice softness”

but ice sheets are shallow

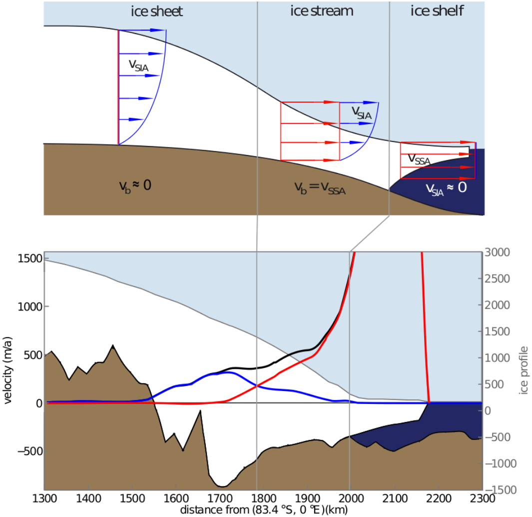
- consider cross section of Greenland ice sheet at 71° N
    - green and blue: usual vertically-exaggerated version



- in red: a view without this vertical exaggeration
- *thus*:
    - most simulations use shallow limits of Stokes
    - high aspect-ratio elements endanger Stokes solvers

sheets versus streams versus shelves

- non-sliding portions of ice sheets flow by shear deformation
- ice streams slide
- "ice shelves" are floating thick ice
- ice shelves flow by extension
  - "membrane" or "plug" flow
- "SIA" and "SSA" will be explained later



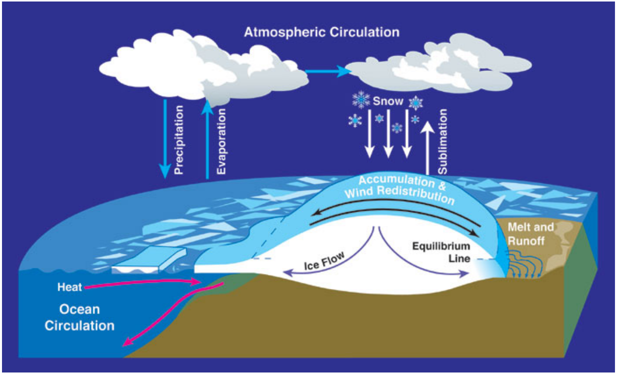
[Lambert glacier and Amery ice shelf, Antarctic]

summary so far

- ice sheets have four outstanding properties *as viscous flows*:
    1. slow
    2. shear-thinning
    3. shallow
    4. contact slip

big picture: ice sheet flow affects sea level

- *mass and energy inputs*: (1) snow adds, (2) sun heats, (3) ocean heats, (4) earth heats
- *mass outputs*: (1) surface meltwater, (2) basal meltwater, (3) ice discharge



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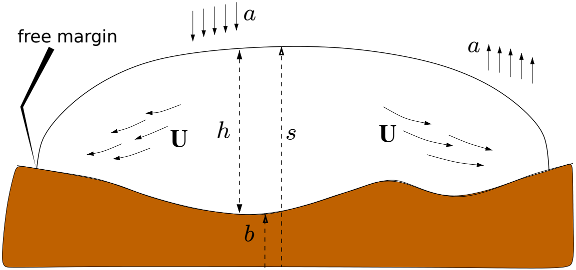
the main variables



- $a(t, x, y, z) =$ yearly-average mass balance
- $b(x, y) =$ bedrock elevation
- $s(t, x, y) =$ ice surface elevation
- $h(t, x, y) =$ ice thickness $= s - b$
- $\mathbf{U}(t, x, y, z) =$ horizontal velocity field

key idea: ice surface s is always above the bedrock b

shallow ice approximation (SIA)

- SIA = lubrication approximation of Stokes model
- good approximation when:
  - sliding is small or zero
  - bedrock slope is modest
- derive SIA equations by scaling Stokes:
  - $[h]$ is a typical thickness scale
  - $[x]$ is a typical width scale
  - small parameter is $\epsilon = [h]/[x]$

movie of time-dependent SIA

- at right is the Halfar similarity solution
- an exact, time-dependent, zero mass balance solution where the $t \to 0^+$ limit is a delta function
- compare Barenblatt solution of porous medium equation

frames from $t = 4$ months to $t = 10^6$ years, equal spaced in *exponential* time

SIA: velocity

- let $p = n + 1 > 2$
- assume: no sliding and isothermal
- horizontal ice velocity is given by:

$$\mathbf{U} = -\frac{2A}{p}(\rho g)^{p-1}\left[(s-b)^p - (s-z)^p\right]|\nabla s|^{p-2}\nabla s$$

- no PDE needs to be solved to compute velocity!

SIA: steady state

- mass conservation in steady state:

$$\nabla \cdot \left(\int_b^s \mathbf{U} \, dz \right) = a$$

- shallow ice approximation + (steady) mass conservation:

$$-\nabla \cdot \left(\Gamma (s-b)^{p+1} |\nabla s|^{p-2} \nabla s \right) = a$$

  - this is the major SIA equation (…a PDE?)
  - computes ice surface s
  - constant $\Gamma > 0$ combines ρ, g, A, p
  - p-Laplacish …but coefficient $(s-b)^{p+1} \to 0$ at margins

a change of variable

- using the change of variable $u = h^{\frac{2p}{p-1}}$, the steady SIA equation is:

$$-\nabla \cdot \left(\mu |\nabla u - \Phi(u)|^{p-2} (\nabla u - \Phi(u))\right) = \alpha(u)$$

  where
  - $\mu > 0$ is constant (isothermal case)
  - $\Phi(u) = -C\, u^{\frac{p+1}{2p}} \nabla b$ is transformed bedrock topography
  - $\alpha(u) = a(x, y, z = u^{\frac{p-1}{2p}})$ is transformed mass balance
- a generalized p-Laplace equation with added nonlinearities
- change of vars means uniform p-ellipticity recovered, but at cost of "tilt" $(\nabla u - \Phi(u))$

SIA: weak formulation = variational inequality

- issue: SIA equation applies only on domain where $s > b \iff h > 0$
- the change $h \to u$ transforms constraint $s \geq b$ into $u \geq 0$
- define convex constraint set

$$\mathcal{K} := \{v \in W_0^{1,p}(\Omega), v \geq 0\}$$

definition

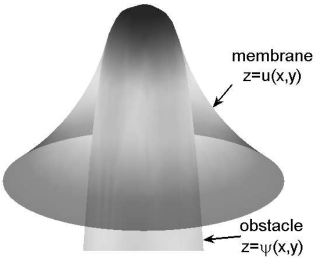
$u \in \mathcal{K}$ solves the *steady shallow ice sheet problem* if

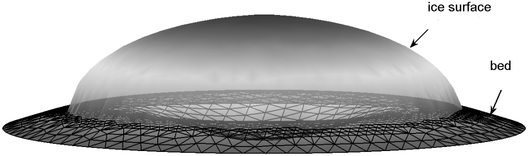
$$\int_\Omega \left(\mu|\nabla u - \Phi(u)|^{p-2}(\nabla u - \Phi(u))\right) \cdot \nabla(v-u) \geq \int_\Omega \alpha(u)(v-u)$$

for all $v \in \mathcal{K}$ (Jouvet-Bueler 2012)

SIA: an analogy

- ice sheet surface = membrane
- bedrock = obstacle





existence and uniqueness for a restricted problem

(easy) theorem

if α, Φ were independent of u then the variational inequality is equivalent to:

$$u \text{ minimizes} \qquad J(v) = \frac{\mu}{p}\int_\Omega |\nabla v - \Phi|^p - \int_\Omega \alpha v$$

over $v \in \mathcal{K}$; this admits a unique solution (Jouvet-Bueler 2012)

- gives ice sheet existence and uniqueness only if
  - bedrock is flat ($\Phi = 0$) and
  - mass balance is elevation-independent ($a = a(x, y)$)
- but otherwise: α, Φ are not independent of u

existence in the general case

- $p > 2$ so $W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$
- define map $\mathcal{A} : C(\overline{\Omega}) \to C(\overline{\Omega})$, which takes w to the unique u solving (over $v \in \mathcal{K}$)

$$\int_\Omega \mu \left(|\nabla u - \Phi(w)|^{p-2} (\nabla u - \Phi(w)) \right) \cdot \nabla(v - u) \geq \int_\Omega \alpha(w)(v - u)$$

result

the map $\mathcal{A}$ admits at least one fixed point (Jouvet-Bueler 2012)

sketch of proof:

- $\mathcal{A}$ is continuous and compact
- the set $\{w \in C(\overline{\Omega}), \exists \lambda \in [0,1]$ so that $w = \lambda \mathcal{A}(w)\}$ is bounded
- Schaefer's fixed point theorem

thus: fixed-point iteration on variational inequality

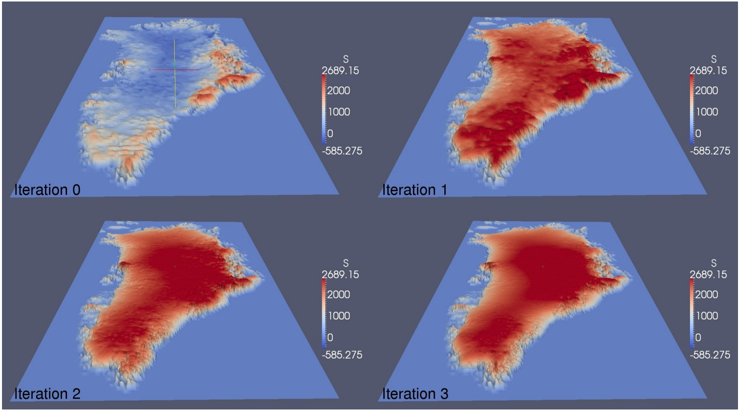
- given bedrock topography $b(x, y)$
- given mass-balance $a(x, y)$ (steady climate)
- set $u_0 = 0$
- do fixed point iterations for $u_{k+1} \in \mathcal{K}$:

$$\int_\Omega \left(\mu|\nabla u_{k+1} - \Phi(u_k)|^{p-2}(\nabla u_{k+1} - \Phi(u_k))\right) \cdot \nabla(v - u_{k+1}) \geq \int_\Omega \alpha(v - u_{k+1})$$

- *computes*: steady state ice sheet shape

example: steady ice sheet on Greenland bedrock

- first iterations from zero ice:

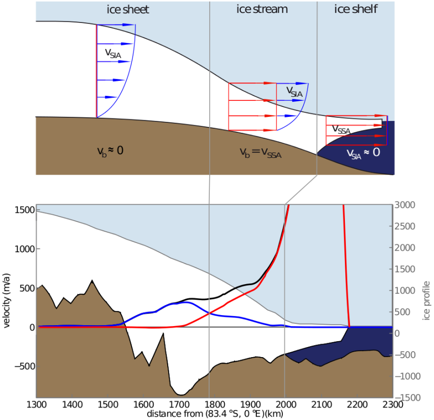


- as far as we can tell: this 2011 computation was the first of the steady state of a real ice sheet *without* time-stepping

Outline

shallow shelf approximation (SSA): a "definition"

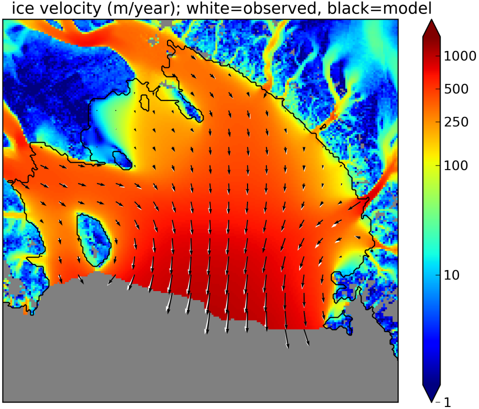
- SSA = "membrane" or "plug" flow approximation of Stokes
- a good approximation when there is low basal resistance and minimal basal topography
- a very good approximation for ice shelves (next slide)
- derived by scaling with $\epsilon = [h]/[x]$ *and* requirement that basal resistance is low (see Schoof (2006))

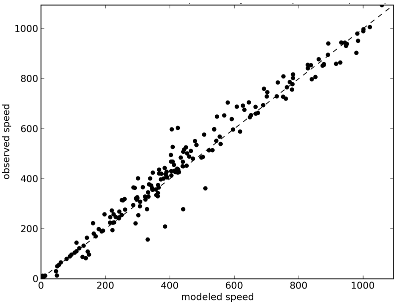


[Lambert glacier and Amery ice shelf, Antarctic]

SSA works *well* for ice shelves

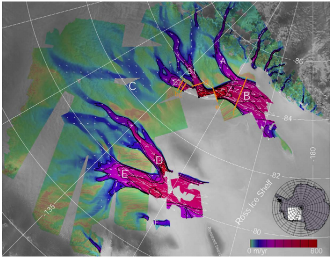
- Ross ice shelf (Antarctica) velocity below
  - observed versus computed by SSA model in PISM
  - tuned: single, constant A

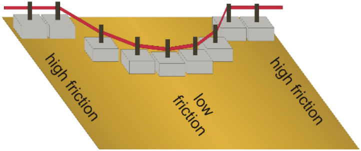




SSA for ice streams: an analogy

- ice shelves have zero basal resistance
- ice streams emerge where basal resistance is sufficiently low (*top*: Siple coast ice streams)
- a basal resistance model:
    - "plastic" or Coulomb friction
    - distribution of yield stress τ_c
- ice membrane connects to upstream and/or lateral high friction with viscous stresses (*bottom*: Schoof's slider analogy)





SSA weak formulation

- let $q = 1 + \frac{1}{n}$ and $B = A^{1-q}$
- suppose a basal yield stress distribution $\tau_c(x,y)$, zero on ice shelves
- $2\,\|\mathbf{V}\|^2 := \sum_{i,j}(\mathbf{D}V_{ij})^2 + \sum_i (\mathbf{D}V_{ii})^2$
- $\mathbf{F}$ denotes lateral force along calving front

definition

the horizontal velocity $\mathbf{U} \in W^{1,q}(\Omega)$ solves the coulomb friction SSA if it minimizes

$$\mathcal{J}_{\mathsf{SSA}}(\mathbf{V}) = \int_\Omega \frac{2B}{q} h\,\|\mathbf{V}\|^q + \rho g h \nabla s \cdot \mathbf{V} + \tau_c|\mathbf{V}| - \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{V}$$

SSA weak formulation is well-posed

Theorem

if $h \in L^\infty(\Omega)$ with $h \geq h_0 > 0$, and if $h|\nabla s| \in L^{q/(q-1)}(\Omega)$, and if $\tau_c \in L^{q/(q-1)}(\Omega)$, and as long as there is sufficient total basal resistance,* then the Coulomb friction SSA is well-posed problem for computing the velocity $\mathbf{U} \in W^{1,q}(\Omega)$ (Schoof, 2006)

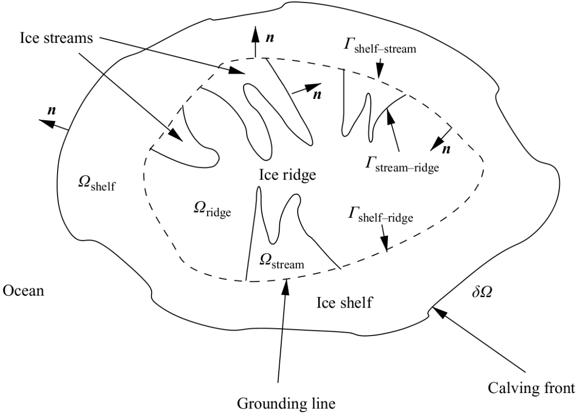
- *note*: because $\mathcal{J}_{\text{SSA}}$ is not differentiable, minimization on last slide is equivalent to a variational inequality but not to a PDE

$*$: To stop the ice sheet from sliding whole into the sea. There is a precise inequality.

Outline

marine ice sheets

- marine ice sheets have all modes of flow
- full of free boundaries
- the Antarctic ice sheet is the marine ice sheet



cartoon from (Schoof, 2006)

ice sheet geometry evolution: set-up

- recall $n \approx 3$ (i.e. $n > 1$):
  - $p = n + 1 > 2$ is for SIA weak formulation
  - define $r = \frac{p-1}{2p}$; SIA change of variables is $u = h^r$
  - $q = 1 + \frac{1}{n} < 2$ is for SSA weak formulation
- time-discretization t_k with spacing $\tau_k = t_{k+1} - t_k$
- time-dependent mass conservation:

$$\frac{\partial h}{\partial t} + \nabla \cdot \left(\int_b^s \mathbf{U}\, dz \right) = a$$

- we hybridize: (Bueler & Brown, 2009)

$$\mathbf{U} = \mathbf{U}_{\text{SIA}} + \mathbf{U}_{\text{SSA}}$$

ice sheet geometry evolution: a new algorithm

1. find velocity $\mathbf{U}_k \in W^{1,q}(\Omega)$ that minimizes

$$\mathcal{J}_{\mathsf{SSA}}(\mathbf{V}) = \int_\Omega \frac{2B}{q} h_k \, \|\mathbf{V}\|^q + \rho g h_k \nabla s_k \cdot \mathbf{V} + \tau_c |\mathbf{V}| - \int_{\partial\Omega} \mathbf{F}_k \cdot \mathbf{V}$$

2. find $h_{k+\frac{1}{2}}$, the solution at t_{k+1} of the advection problem:

$$\begin{cases} \frac{\partial h}{\partial t} + \nabla \cdot (h \, \mathbf{U}_k) = 0, & t_k \le t \le t_{k+1}, \\ h(t_k) = h_k. \end{cases}$$

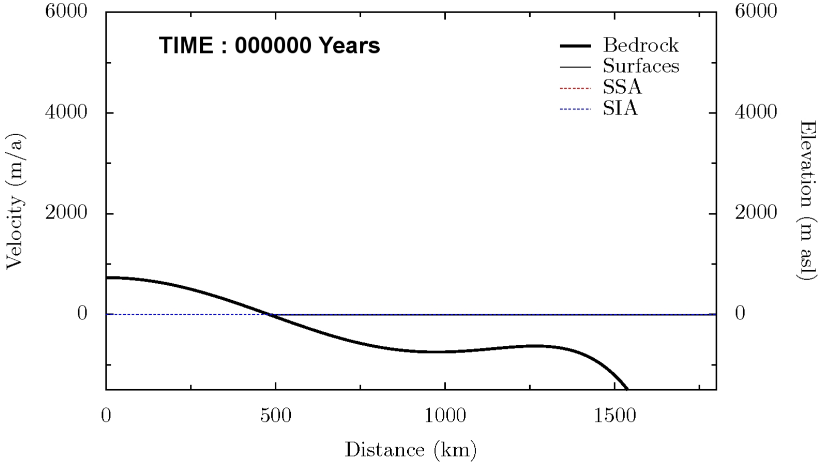
3. transform: $u_{k+\frac{1}{2}} = (h_{k+\frac{1}{2}})^{1/r}$
4. find thickness $h_{k+1} = u^r$, i.e. find $u \in \mathcal{K}$, that minimizes

$$\mathcal{J}_{\mathsf{SIA}}(v) = \int_\Omega \frac{1}{(r+1)\tau_k} v^{r+1} + \frac{\mu}{p} |\nabla v - \Phi(u_{k+\frac{1}{2}})|^p - \left(\frac{1}{\tau_k} u_k^r + \alpha(u_{k+\frac{1}{2}}) \right)$$

5. repeat at 1.

(Jouvet et al. to appear)

moving grounding line movie



numerical solution of the weak formulations

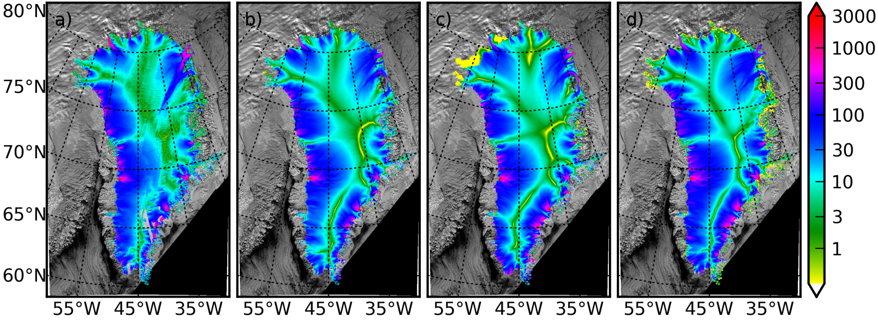
- the $W^{1,p}$ (SIA) and $W^{1,q}$ (SSA) solutions to these free boundary problems have low regularity
  - so we use P_1 finite elements
  - $\|u - u_{\mathsf{h}}\|_{W^{1,p}} \le C\mathsf{h}^{2/p}$: convergence is slow
- previous movie used:
  - Truncated Nonsmooth Newton MultiGrid (TNNMG) method for both $\mathcal{J}_{\mathsf{SSA}}$ and $\mathcal{J}_{\mathsf{SIA}}$ minimizations
  - implemented in DUNE (`dune-project.org`)

known concerns with algorithm

- $\mathcal{J}_{\mathsf{SSA}}$ needs regularization so that h_k is lower bounded
- advection scheme should maintain $h \geq 0$
  - for now: first-order upwinding on advection problem
- first-order time-splitting

results from PISM

- PISM = Parallel Ice Sheet Model (`pism-docs.org`)
- below are 2 km grid results for Greenland; everything evolves; only showing surface velocities
- PISM is "old technology": implements SIA+SSA hybrid but in strong form with ad hoc treatment of free boundaries



a) observed; b) constant-climate; c) paleo-climate; d) flux-corrected

(computations by Andy Aschwanden)

conclusion

some new thinking which is weak and shallow

- steady grounded ice sheets now have a (mostly) well-posed shallow, weak, obstacle-like formulation (SIA)
- sliding velocity computations are by a shallow weak formulation in which ice streams “emerge naturally” (SSA)
- both of above are generalizations of p-Laplace problems
- new marine ice sheet algorithm from time-splitting:
  - solve SSA weak form
  - advection with SSA velocity
  - solve SIA+(mass conservation) obstacle problem

a quality of the SIA variational inequality

- every glaciologist believes this about steady climates:

  if $a > 0$ on a sub-domain R then $s > b$ on R

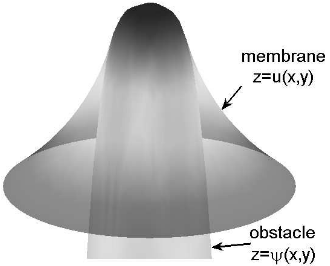
- that is:

  if it snows more than it melts then you get a glacier there

- uniformly-elliptic variational inequalities, e.g. the classical obstacle problem,

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u),$$

  for all $v \geq \psi$, do *not* have the analogous property



on TNNMG

to minimize a constrained or non-differentiable functional $\mathcal{J}$:

- let I be the entire node index set, $\mathcal{I} = \mathcal{I}(v)$ the active set where v is away from the obstacle/non-differentiability
- let $\mathcal{F} : \mathbb{R}^I \to \mathbb{R}^I$ be a "nonlinear Gauss-Seidel smoother":
  - gives correction that minimizes $\mathcal{J}$ at each node separately
  - can be inexact
  - the active set $\mathcal{I}$ can change
- let $\mathcal{D}$ be the domain of $\mathcal{J}$ and $\mathcal{P}_{\mathcal{D}}$ be a projection onto $\mathcal{D}$
- then TNNMG generates sequence u^l by:

$$
\begin{aligned}
u^{l+\frac{1}{3}} &= u^l + \mathcal{F}(u^l),\\
u^{l+\frac{2}{3}} &= u^{l+\frac{1}{3}} - \left(\mathcal{J}''(u^{l+\frac{1}{3}})_{\mathcal{I},\mathcal{I}}\right)^{-1} \mathcal{J}'(u^{l+\frac{1}{3}})_{\mathcal{I}},\\
u^{l+1} &= \operatorname{argmin}_{w,\rho\in[0,1]} \left\{ \mathcal{J}(w) \middle| w = \rho u^{l+\frac{1}{3}} + (1-\rho)\mathcal{P}_{\mathcal{D}}(u^{l+\frac{2}{3}}) \right\}
\end{aligned}
$$