Implicit time-stepping for ice sheets

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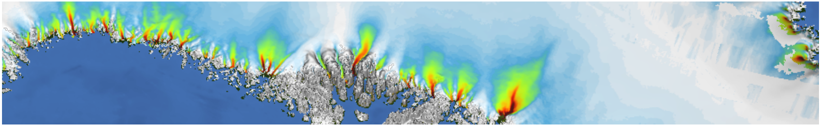

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Outline

problem, goals, and models

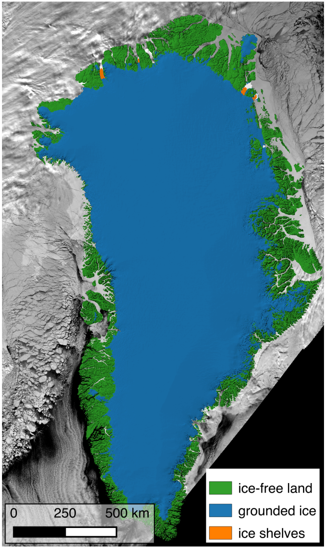
semi-discretizations

solving the equations for one time step

some early results

ice sheet flows and their boundaries

- nearly-fractal free boundaries
    - location determined by flow, topography, and atmosphere/ocean inputs
- large fraction of the boundary: grounded margins
- surface slope is discontinuous at grounded margins



PISM mask, by A. Aschwanden

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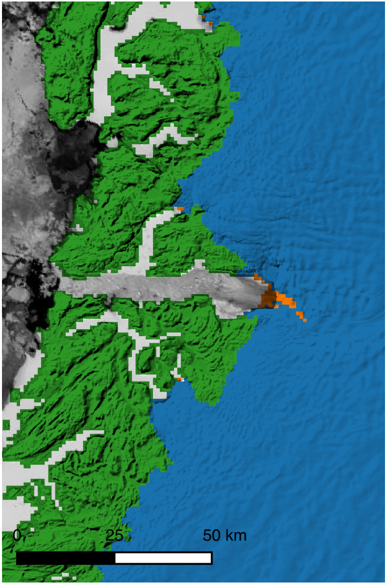

ice sheet flows and their boundaries

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  - location determined by flow, topography, and atmosphere/ocean inputs
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Taku Glacier, Alaska, by M. Truffer

computational goal: long-time, high-res runs

- my goal:

  routinely simulate ice sheets on their natural time scales (ice age cycles $\gtrsim 10^5$ years) at resolutions where bedrock bumps, outlet glaciers, ice streams, and grounding lines are resolved ($\lesssim 500$ m)

- PISM (Parallel Ice Sheet Model) not there yet . . . either long-time *OR* high-res
- some concerns:
  1. stress-balance solves expensive
     - if time steps are short then stress balance is a lot of work
     - . . . energy conservation (temperature and basal melt) too!
  2. evolution of ice thickness $H(t, x, y)$ is diffusive . . . thus stiff
     - because ice flows downhill
  3. ice margins are low-regularity
     - 2 reasons: (*i*) constraint $H \geq 0$ and (*ii*) degeneracy
  4. bedrock is steep

a performance model

- grid spacing $h = \Delta x = \Delta y$ in 2D
  - (degrees of freedom) $= O(h^{-2})$
- time step limited by stability or accuracy:

$$\Delta t \le O(h^q)$$

  - $q = 2$ for conditionally-stable explicit schemes on diffusions
  - accuracy alone suggests $0 < q < 1$? ...a scientific question?
- solution at one time step:
  - $N(h)$ Newton iterations
  - $K(h)$ (preconditioned) Krylov steps per Newton
- cost of computation on $\Omega \times [0, T]$:

$$\begin{aligned} C(h) &= \text{(number of steps)} \cdot \text{(iterations per step)} \cdot \text{(cost of 1 residual)} \\ &= O(h^{-q}) \cdot N(h) \cdot K(h) \cdot O(h^{-2}) \end{aligned}$$

  - explicit: $C(h) = O(h^{-2}) \cdot 1 \cdot 1 \cdot O(h^{-2}) = O(h^{-4})$ $\leftarrow$ beat this!

ice sheet models

- fixed computational domain $\Omega \subset \mathbb{R}^2$ where inputs $b =$ (bed elevation) and $m =$ (mass balance) are given
    - Ω is only partly-covered by ice
- shallow, possibly-hybrid, thickness-based $\qquad H,\ \mathbf{u} = (u, v)$

$$H_t + \nabla \cdot (-D \nabla H + \overline{\mathbf{u}} H) = m(x,t) \qquad \text{mass conservation}$$
$$\mathcal{L}(\mathbf{u}, H) = 0 \qquad \text{shallow stress balance}$$

- *also*: Stokes, surface-elevation-based $\qquad s,\ \mathbf{u} = (u, v, w)$

$$s_t + \mathbf{u}\big|_s \cdot (s_x, s_y, -1) = m(x,t) \qquad \text{surface kinematical}$$
$$\nabla \cdot \mathbf{u} = 0 \qquad \text{incompressibility}$$
$$\mathcal{L}(\mathbf{u}, s) = 0 \qquad \text{Stokes stress balance}$$

- notation & assumptions:
    - H thickness, s surface elevation, $\mathbf{u}$ velocity
    - $D = D(H, |\nabla s|)$ is SIA diffusivity (nonlinear & degenerate)
    - conservation of energy ignored for simplicity
    - Eulerian, fixed grid (structured or not)

semi-discretize in space

- method of lines (MOL)
  - *can you hand the thing to an ODE solver?*
- well-known: MOL for slow fluids is a DAE problem

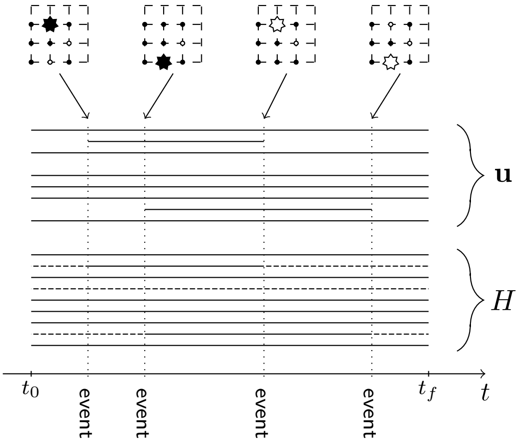
$$\begin{aligned} \dot{H} &= f(H, \mathbf{u}, t) && \text{mass conservation} \\ 0 &= g(H, \mathbf{u}) && \text{momentum conservation} \end{aligned}$$

  - *isn't implicit time-stepping required for DAEs anyway?*
- velocity variables only exist at positive-thickness locations i:

$$\mathbf{u}_i \text{ exists} \quad \Longleftarrow \quad H_i > 0$$

- thus ODE solver must handle *events* like these?:
  - ice disappears during time step: $H_i(t) > 0 \to H_i(t + \Delta t) = 0$
  - ice appears during time step: $H_i(t) = 0 \to H_i(t + \Delta t) > 0$

MOL+events cannot scale



- at each event the ice velocity dimension changes
- ice sheet margins nearly fractal, so *a lot* of events to handle
- re-meshing at *every* event probably won't scale

semi-discretize in time

- semi-discretize in time *for understanding*
- consider a single backward Euler time-step
  - better time-stepping later
- hybrid equations become (notation: $H = H_{\text{new}}, \mathbf{u} = \mathbf{u}_{\text{new}}$):

$$\begin{aligned} H - H_{\text{old}} + \Delta t\, \nabla \cdot (-D \nabla H + \mathbf{u} H) &= \Delta t\, m \\ \mathcal{L}(\mathbf{u}, H) &= 0 \end{aligned}$$

single time-step problem for mass conservation

- solve for H subject to $H \geq 0$:

$$H - H_{\text{old}} + \Delta t\, \nabla \cdot \mathbf{q} = \Delta t\, m$$

  - where $\mathbf{q} = -D \nabla H + \mathbf{u} H$
  - note: $D = D(H, |\nabla s|) \to 0$ at margins

- make rigorous two ways:
  - variational inequality (VI)

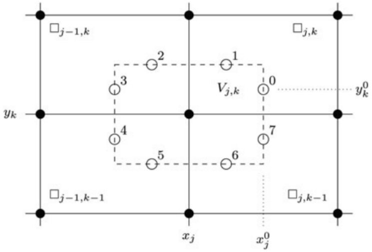
$$\int_\Omega (H - H_{\text{old}} - \Delta t\, m)(\eta - H) - \Delta t\, \mathbf{q} \cdot \nabla(\eta - H) \geq 0, \quad \forall \eta \geq 0$$

  - nonlinear complementarity problem (NCP)

$$\begin{aligned} F(H) &= H - H_{\text{old}} + \Delta t\, \nabla \cdot \mathbf{q} - \Delta t\, m \geq 0 \\ H &\geq 0 \\ H\, F(H) &= 0 \end{aligned}$$

solving the equations

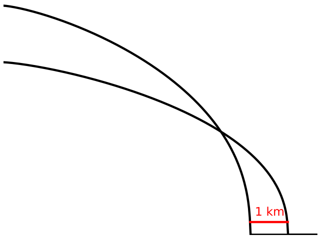
- discretize in space:
  - e.g. $M^\star = (Q^1$ elements and *finite volume* weak form "$\int_V$") Bueler 2016

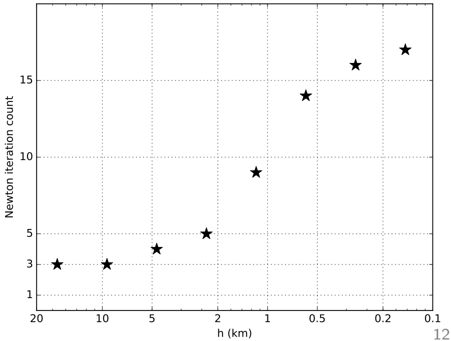


- wrote small PETSc code for mass conservation problem:
  - MOL using $M^\star$
  - tells TS object which part is stiff: $F(H_t, H) = G(t, H)$
    - $F = H_t + \nabla \cdot \mathbf{q}$ and $G = m$
  - allows any implicit or IMEX time-stepping
- equations at each time step are solved with
  - NCP-adapted ("reduced-space" or "semi-smooth") Newton
    - `-snes_type vinewton{rs|ss}ls` Benson & Munson 2006
  - + Krylov solver + preconditioning

margin advance experiment: Newton iterations

- consider a single 10-year `beuler` time step
- of a Greenland-sized radial ice sheet
  - flat bed, $m = 0$
  - margin advance 975 m
- reduced-space Newton solver sees Jacobian in inactive variables only
  - states are admissible
  - dimension changes at each Newton step
- on fine grids ($\lesssim 1$ km) the number of Newton iterations is proportional to margin motion divided by Δx





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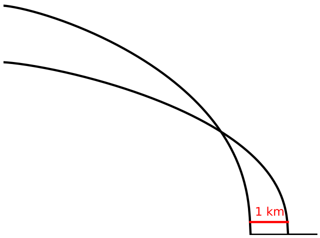
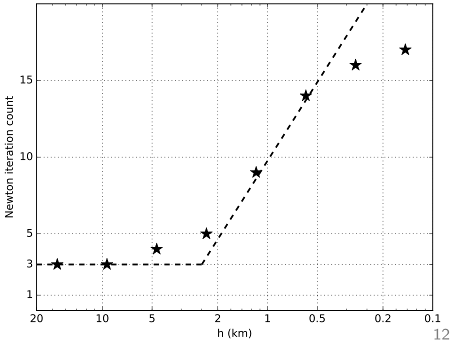
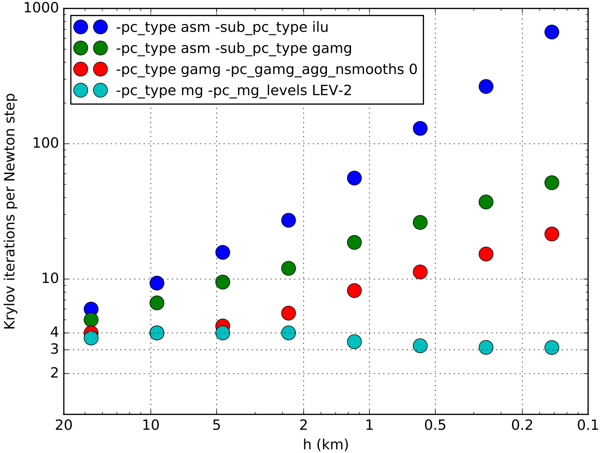




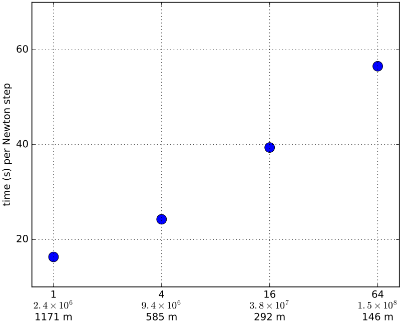
margin advance experiment: preconditioners

- fixed MPI rank = 64
- compare preconditioners:
  `-snes_type vinewtonrsls -ksp_type gmres -pc_type X`
- Krylov iterations per Newton step:



margin advance experiment: weak scaling?

- very preliminary weak-scaling evidence
  - ranks $1, 4, 16, 64$
  - fixed d.o.f. per process: 2.4×10^6
  - `-pc_type mg`
- time per Newton step: . . . *should be flat!*



- the good news: $\Delta t / \Delta t_{FE} = 9 \times 10^4$ on finest grid

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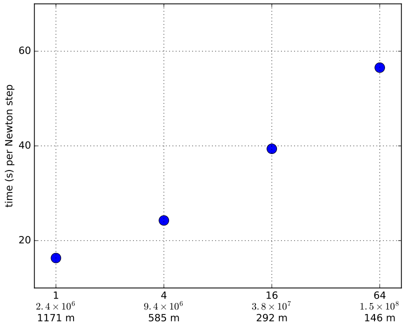

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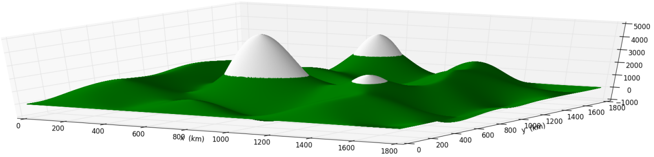
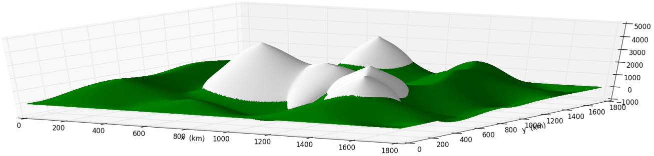
example: 50ka run with topography and sliding



$t = 0$ a

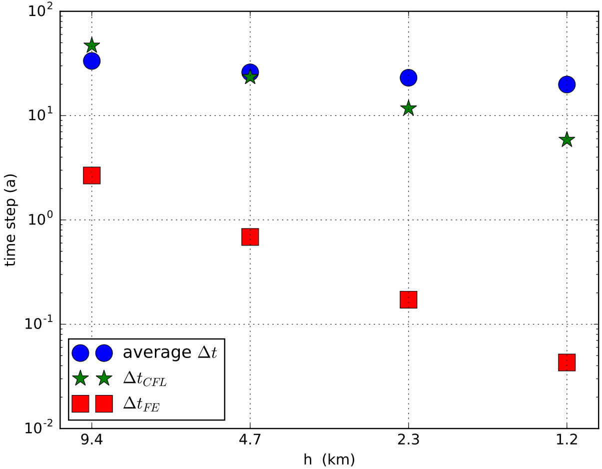


$t = 50$ ka

- solve: $H_t + \nabla \cdot \mathbf{q} = m$ where $\mathbf{q} = -D\nabla H + \mathbf{u}H$
  - imposed "sliding" $\mathbf{u}(x, y)$
  - elevation-dependent accumulation/ablation $m = m(s)$

50ka run: time-stepping performance

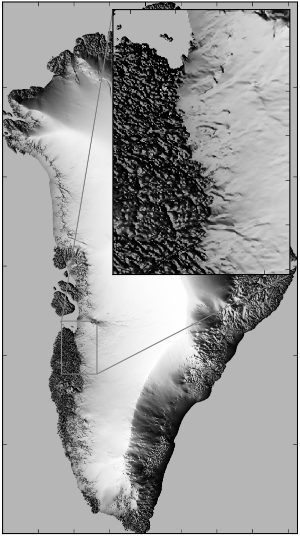
- ARKIMEX(3): adaptive Runge-Kutta implicit/explicit 3rd-order (3 stage) time-stepping
- at least three nonlinear solves per time step



example: Greenland ice sheet $\Delta t = \infty$

- steady geometry $H(x, y)$ of the Greenland ice sheet
  - given $m(x, y)$ and $b(x, y)$
- Bueler 2016, J. Glaciol.
- one $\Delta t = \infty$ step
  - 900 m structured grid
  - 7×10^6 d.o.f.

- *but* Newton convergence suffers from bed roughness



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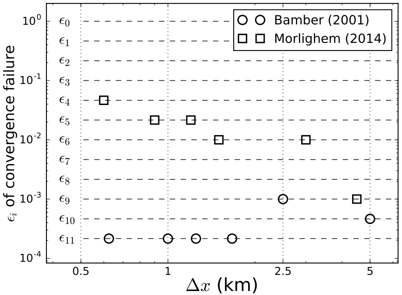

summary

- recommendations for implicit time-stepping of thickness-based mass conservation:
  - enforce $H \geq 0$ as NCP or VI
  - use reduced-space solver which has admissible states for stress balance solution
  - use geometric multigrid (?)
  - result: $> 10^5 \, \Delta t_{\mathsf{FE}}$ achievable
- some limitations:
  - extra Newton steps needed to move margins x grid spaces
  - bed roughness eventually limits Newton solver convergence
  - calving and front-melting not addressed in this framework

    . . . yet

- wiser now? . . . if I were to start over with PISM:

```
mpiexec -n N newpism -da_refine M \
   -ts_type arkimex -snes_type vinewtonrsls -pc_type mg
```