Free-boundary problems in models of the cryosphere

Ed Bueler

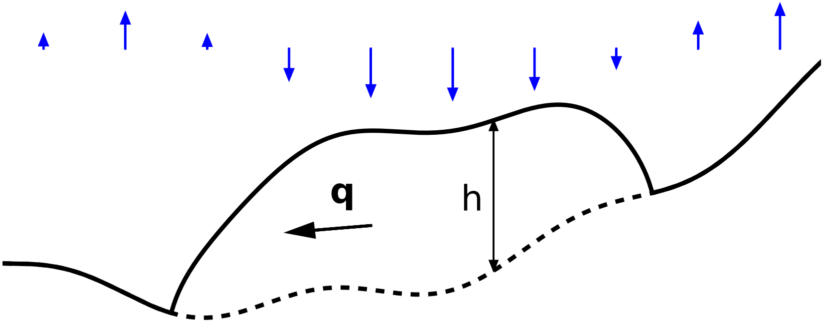
Dept of Mathematics and Statistics, and Geophysical Institute
University of Alaska Fairbanks
(*funded by NASA Modeling, Analysis, and Prediction program*)

SIAM GS July 2015
Stanford University

Outline

1. The problem I'm worried about:
   - time-stepping free-boundary fluid layer models
2. Practical conclusions:
   - approach I: semi-discretize in time
   - approach II: each time-step is weakly-posed free-bdry problem
   - newly-available numerical tools
   - new (?) limitations to discrete conservation

A fluid layer in a climate

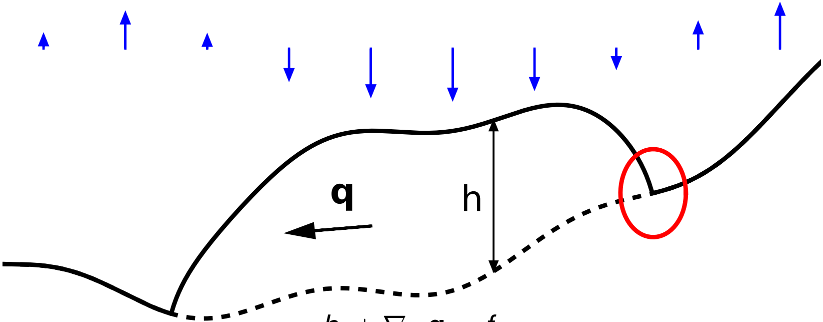


- mass conservation PDE for a layer:

$$h_t + \nabla \cdot \mathbf{q} = f$$

  - h is a thickness: $h \geq 0$
  - mass conservation PDE applies only where $h > 0$
  - $\mathbf{q}$ is flow (vertically-integrated)
  - source f is "climate"; $f > 0$ shown downward

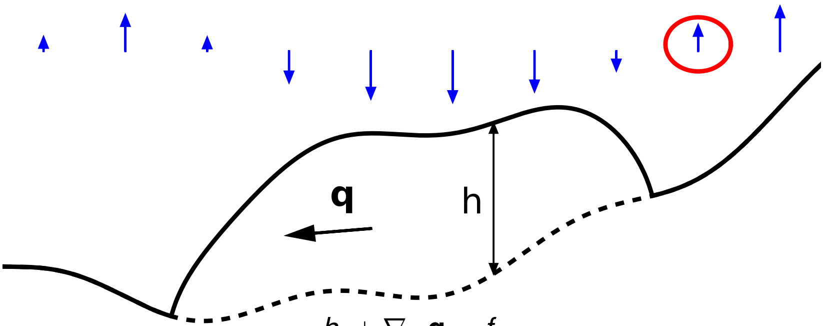
A fluid layer in a climate: *the troubles*



$$h_t + \nabla \cdot \mathbf{q} = f$$

- $h = 0$ *and what else* at free boundary?
  - shape at free boundary depends on both $\mathbf{q}$ and f
- $f < 0$ not "detected" by model where $h = 0$
  - how to do mass conservation accounting?
- $f \approx 0$ threshold behavior
  - $h > 0$ as soon as $f < 0$ switches to $f > 0$

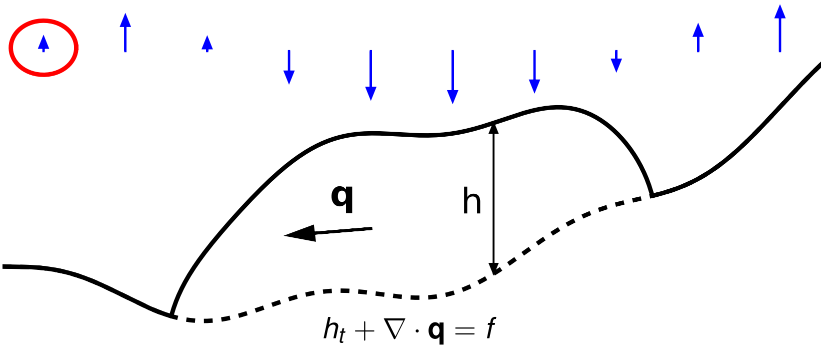
A fluid layer in a climate: *the troubles*



$$h_t + \nabla \cdot \mathbf{q} = f$$

- $h = 0$ *and what else* at free boundary?
  - shape at free boundary depends on both $\mathbf{q}$ and f
- $f < 0$ not "detected" by model where $h = 0$
  - how to do mass conservation accounting?
- $f \approx 0$ threshold behavior
  - $h > 0$ as soon as $f < 0$ switches to $f > 0$

A fluid layer in a climate: *the troubles*



- $h = 0$ *and what else* at free boundary?
  - shape at free boundary depends on both $\mathbf{q}$ and f
- $f < 0$ not "detected" by model where $h = 0$
  - how to do mass conservation accounting?
- $f \approx 0$ threshold behavior
  - $h > 0$ as soon as $f < 0$ switches to $f > 0$

Examples

glaciers

ice shelves & sea ice

tidewater marsh

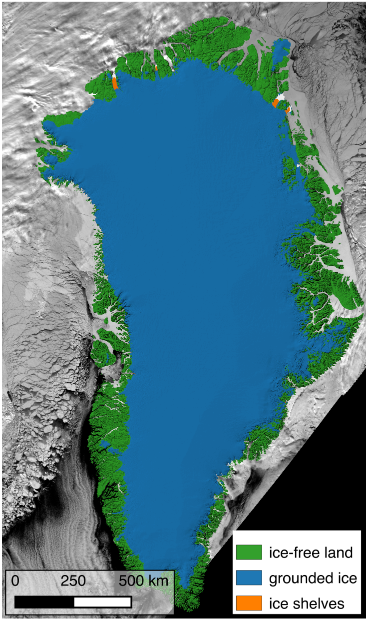
tsunami inundation

and also surface hydrology, subglacial hydrology, . . .

I'm driven here by practical modeling: ice sheets

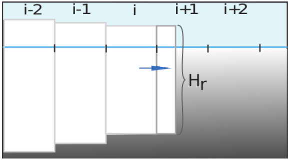
- the icy region is nearly-fractal and disconnected
- currently in our ice sheet model*:
  - explicit time-stepping
  - free boundary by truncation
- want for our model:
  - long implicit time steps
  - better conservation accounting to user

*Parallel Ice Sheet Model, `pism-docs.org`

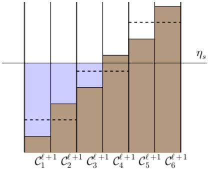


Has anyone solved these problems before?

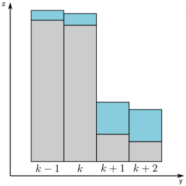
- yes, of course!
- generic result: *ad hoc* schemes near the free boundary



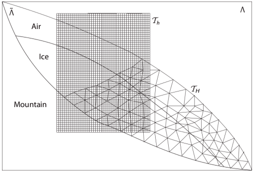
volume-of-fluid method at ice shelf fronts
(Albrecht et al, 2011)



tsunami run-up on shore
(LeVeque, George, Berger, 2011)



glacier ice on steep terrain
(Jarosch, Schoof, Anslow, 2013)



volume-of-fluid method at glacier surface
(Jouvet et al 2008)

Approach I: semi-discretize in time

$$h_t + \nabla\cdot\mathbf{q} = f \qquad \to \qquad \frac{H_n - H_{n-1}}{\Delta t} + \nabla\cdot\mathbf{Q}_n = F_n$$

- semi-discretize in time: $H_n(x) \approx h(t_n, x)$
- the new equation is strong form single time-step problem
  - a PDE in space where $H_n > 0$
  - details of flux $\mathbf{Q}_n$ and source F_n come from time-stepping scheme
    - e.g. θ-methods or RK

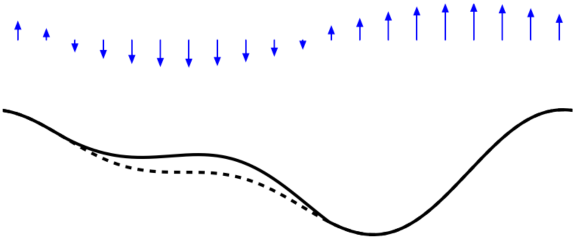
1D time-stepping examples (and my **q**-agnosticism)

same:

- equation

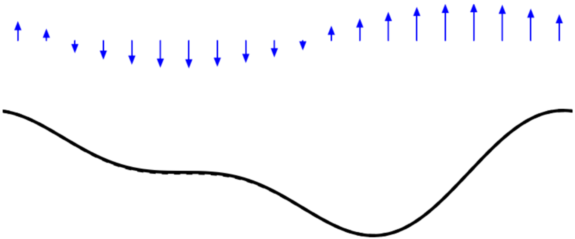
$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = f$$

- BEuler time-step
- climate f
- bed shape
- constraint-respecting Newton scheme



top:

$\mathbf{Q}_n = v_0 H_n$
hyperbolic advection with constant velocity



bottom:

$$\mathbf{Q}_n = -\Gamma |H_n|^{n+2} \cdot |\nabla s_n|^{n-1} \nabla s_n$$

nonlinear degenerate diffusion

Approach II: weak form incorporates $H_n \geq 0$ constraint

- define:

$$\mathcal{K} = \left\{ v \in W^{1,p}(\Omega) \,\middle|\, v \geq 0 \right\} = \text{admissible thicknesses}$$

- we say $H_n \in \mathcal{K}$ solves the weak single time-step problem if

$$\int_\Omega H_n(v - H_n) - \Delta t\, \mathbf{Q}_n \cdot \nabla(v - H_n) \geq \int_\Omega (H_{n-1} + \Delta t\, F_n)\,(v - H_n)$$

  for all $v \in \mathcal{K}$

  - derive this *variational inequality* (VI) from:
    - integration-by-parts on strong form
    - thought about $H_n = 0$ areas

Weak solves strong; gives more info

- assume $\mathbf{Q}_n = 0$ when $H_n = 0$
  - this means $\mathbf{Q}_n$ describes a *layer*
- if $H_n \in \mathcal{K}$ solves weak single time-step problem (VI) then
  - PDE applies on the set where $H_n > 0$ (interior condition):

$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

  - plus inequality on the set where $H_n = 0$:

$$H_{n-1} + \Delta t\, F_n \leq 0$$

    - ⋆ "climate negative enough during time step to remove old thickness"

Alternative weak formulation: NCP

- NCP = nonlinear complementarity problem
- abstractly, NCP is:
  - given differentiable map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$
  - solve
$$\mathbf{x} \geq 0, \quad \mathbf{F}(\mathbf{x}) \geq 0, \quad \mathbf{x}^\top \mathbf{F}(\mathbf{x}) = 0$$
- our case:
  - ∞ dimensions with m.c. equation $h_t + \nabla \cdot \mathbf{q} = f$
  - $\mathbf{x} = H_n$ and $\mathbf{F}(\mathbf{x}) =$ (residual from discrete-time m.c. eqn.)
- in finite dimensions we have VI $\leftrightarrow$ NCP equivalence:

$$\langle \mathbf{F}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0 \quad \forall \mathbf{y} \in \mathcal{K} \qquad \iff \qquad \text{NCP}$$

Numerical solution of the weak problem

the weak single time-step problem:

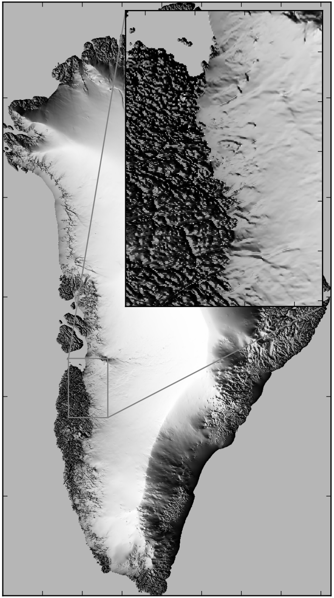
- is nonlinear because of constraint (even for $\mathbf{Q}_n$ linear in H_n)
- can be solved by a Newton method modified for constraint
- scalable implementations are in PETSc* 3.5+
    - see “SNESVI” object
    - for NCP there are two implementations (Benson & Munson, 2006):
        - ⋆ reduced-set (active-set) method
        - ⋆ semismooth method

*Portable Extensible Toolkit for Scientific computation, `www.mcs.anl.gov/petsc`

Example: Greenland ice sheet

- given steady climate and bedrock elevations, what is shape of Greenland ice sheet?
  - climate = "surface mass balance"
    = precipitation − runoff-from-melt
- assume simplest reasonable dynamics: non-sliding shallow ice approximation
- solve VI/NCP weak problem
  - steady state ($\Delta t \to \infty$)
  - reduced-set Newton method
  - 900 m structured grid
  - Q^1 FEs in space
  - $N = 7 \times 10^6$ d.o.f.

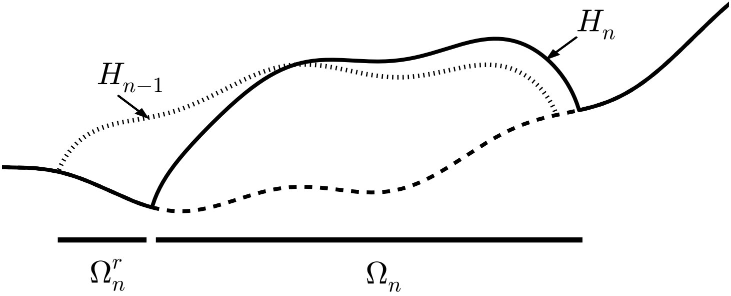
(Bueler, submitted to J. Glaciol.)



Conservation reporting: subsets

- suppose H_n solves the single time-step problem
- define

$$\begin{aligned}\Omega_n &= \{H_n(x) > 0\} \\ \Omega_n^r &= \{H_n(x) = 0 \text{ and } H_{n-1}(x) > 0\} \qquad \leftarrow \text{retreat set}\end{aligned}$$



Conservation reporting: time-series

- define:

$$M_n = \int_\Omega H_n(x)\,dx \qquad \textit{mass at time } t_n$$

- then

$$\boxed{\Delta t\,(-\nabla\cdot\mathbf{Q}_n + F_n)}$$

$$\begin{aligned} M_n - M_{n-1} &= \int_{\Omega_n} \boxed{H_n - H_{n-1}}\,dx + \int_{\Omega_n^r} 0 - H_{n-1}\,dx \\ &= \Delta t\left(0 + \int_{\Omega_n} F_n\,dx\right) - \int_{\Omega_n^r} H_{n-1}\,dx \end{aligned}$$

- new term:

$$R_n = \int_{\Omega_n^r} H_{n-1}\,dx \qquad \textit{retreat loss during step } n$$

Conservation reporting: *limitation*

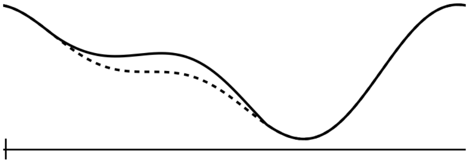
- the retreat loss R_n is not balanced by the climate
  - R_n is *caused* by the climate, but we don't know a computable integral of climate F_n to balance it
- we must track three time series:
  - mass at time t_n: $M_n = \int_\Omega H_n(x)\,dx$
  - climate (e.g. surface mass bal.) over current fluid-covered region:

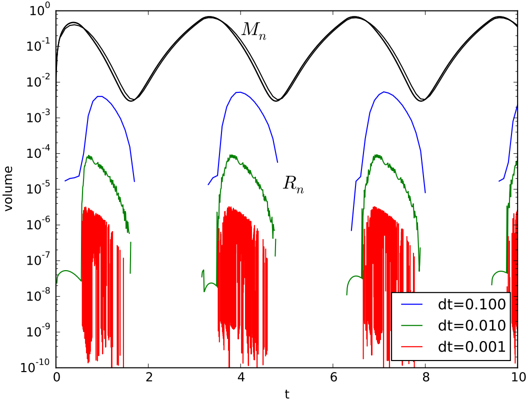
$$C_n = \Delta t \int_{\Omega_n} F_n\,dx \qquad \approx \int_{t_{n-1}}^{t_n} \int_{\Omega_n} f(t,x)\,dx\,dt$$

  - retreat loss during time step: $R_n = \int_{\Omega_n^r} H_{n-1}\,dx$
- now it balances:

$$M_n = M_{n-1} + C_n - R_n$$

Reporting discrete conservation: $R_n \to 0$ as $\Delta t \to 0$





Summary

consider layer flow model $h_t + \nabla \cdot \mathbf{q} = f$ subject to signed climate f and where h is layer thickness

- goals/issues:
  - long time steps wanted
  - models have been limited by free-boundary lack-of-clarity
- approach:
  - include constraint on thickness: $h \geq 0$
  - consider discrete-time problem before doing FEM/FVM/etc.
  - pose single time-step problem weakly as VI or NCP
  - solve by scalable constrained-Newton method (PETSc)
- new (?) result:
  - discrete conservation requires tracking retreat-loss time-series
    - ⋆ in addition to climate input during time step