What is . . . a *variational inequality*?

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what is . . . the source of my title?

WHAT IS . . .

a Gröbner Basis?

Bernd Sturmfels

WHAT IS . . .

a Quasi-morphism?

D. Kotschick

WHAT IS . . .

a Random Matrix?

Persi Diaconis

WHAT IS . . .

a Systole?

Marcel Berger

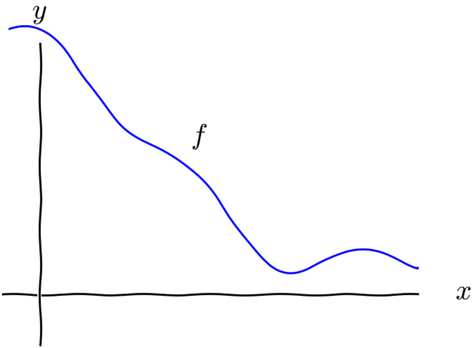
Outline

problems you can write as variational inequalities

obstacle problem example

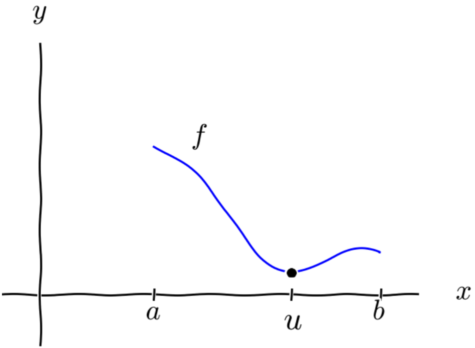
three variational inequalities for glaciers

calculus I problem



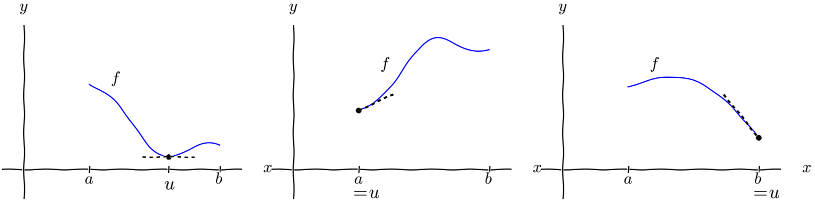
- suppose you have a smooth function
- and you want to minimize it

calculus I problem



- suppose you have a smooth function *on a closed, bounded interval*
- and you want to minimize it

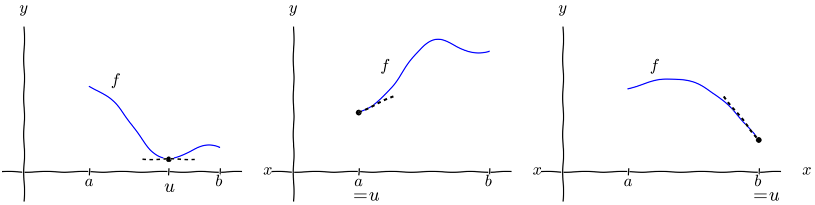
calculus I problem



because f is smooth, you can say about the minimizer u that:

- if $a < u < b$ then $f'(u) = 0$ *or*
- if $u = a$ then $f'(u) \geq 0$ *or*
- if $u = b$ then $f'(u) \leq 0$

calculus I problem



because f is smooth, you can say about the minimizer u that: the variational inequality applies,

$$f'(u)(v-u) \geq 0 \qquad \text{for all } v \in [a,b]$$

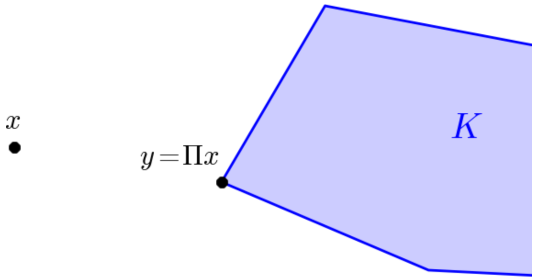
what is a *variational inequality*?

1. cute way to rewrite a calc problem in $\mathbb{R}^1$; min f solves:

$$u: \qquad f'(u)(v-u) \geq 0 \qquad \text{for all } v \in [a,b]$$

the above is a *necessary condition* only

projection onto a closed, convex set $K \subset \mathcal{H}$



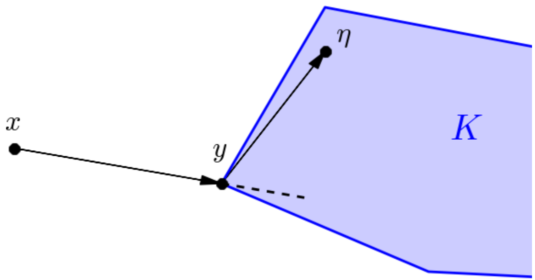
Suppose $K \subset \mathbb{R}^n$ is closed and convex.
(Or $K \subset \mathcal{H}$ closed and convex, where $\mathcal{H}$ is a Hilbert space.)

Definition. Given $x \in \mathbb{R}^n$ (or $x \in \mathcal{H}$), the unique minimizer

$$y = \min_{z \in K} \|x - z\|$$

is the *projection* of x onto K, written $y = \Pi x$.

projection onto a closed, convex set $K \subset \mathcal{H}$



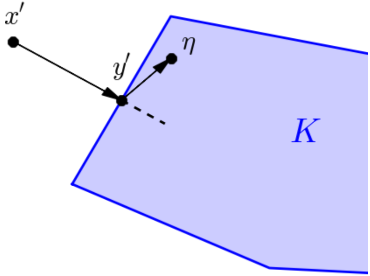
Theorem.

$$y = \Pi x \qquad \iff \qquad (y - x) \cdot (\eta - y) \geq 0 \;\text{ for all } \eta \in K$$

This is also a variational inequality.

Idea. The angle between $y - x$ and $\eta - y$ is $\leq 90^\circ$ for all $\eta \in K$.

projection onto a closed, convex set $K \subset \mathcal{H}$



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$$u: \qquad f'(u)(v-u) \geq 0 \qquad \text{for all } v \in [a,b]$$

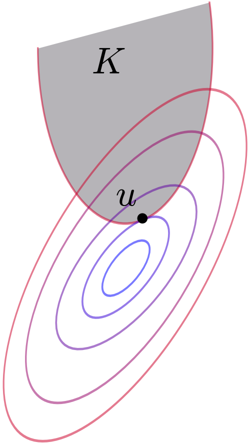
2. dot-product for projection on a closed, convex $K \subset \mathcal{H}$:

$$y = \Pi x: \qquad (y-x)\cdot(\eta - y) \geq 0 \quad \text{for all } \eta \in K$$

minimize f on convex $K \subset \mathbb{R}^n$

Consider a mainstream math problem:

- let $K \subset \mathbb{R}^n$ be convex
- let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth (C^1)
- find $u \in K$ so that f is minimum?



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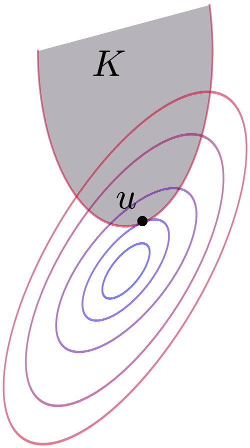


Claim: if $u \in K$ minimizes f then u solves

$$\nabla f(u) \cdot (v - u) \geq 0 \qquad \text{for all } v \in K$$

which is a variational inequality

minimize f on convex $K \subset \mathbb{R}^n$

Theorem. $K \subset \mathbb{R}^n$ convex. $f : \mathbb{R}^n \to \mathbb{R}$ smooth.
If $u \in K$ minimizes f then

$$\nabla f(u) \cdot (v - u) \geq 0 \qquad \text{for all } v \in K.$$

Proof. If $0 \leq t \leq 1$ then

$$(1-t)u + tv \in K$$

But $t = 0$ is minimizer of

$$g(t) = f\left((1-t)u + tv\right)$$

so by the “calculus I problem”, $g'(0) \geq 0$. By chain rule

$$g'(t) = (\nabla f)((1-t)u + tv) \cdot (-u + v)$$

so $g'(0) = (\nabla f)(u) \cdot (v - u) \geq 0$. □

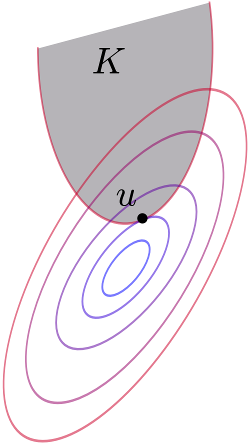
minimize f on convex $K \subset \mathbb{R}^n$

How about existence of u?
It happens if either:

- K is compact
- K is closed and f is coercive

How about uniqueness of u?
It happens if:

- f is strictly convex



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1. cute way to rewrite a calc problem in $\mathbb{R}^1$; min f solves:

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2. dot-product for projection on a closed, convex $K \subset \mathcal{H}$:

$$y = \Pi x: \qquad (y-x)\cdot(\eta - y) \geq 0 \quad \text{for all } \eta \in K$$

3. rewriting of "min f on convex $K \subset \mathbb{R}^n$":

$$u: \qquad \nabla f(u)\cdot(v-u) \geq 0 \qquad \text{for all } v \in K$$

solve $F(x) = 0$ on $\mathbb{R}^n$

Consider another mainstream applied math problem:

- Given continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$, find $x \in \mathbb{R}^n$ so that

$$F(x) = 0.$$

- That is, solve n nonlinear equations in n unknowns.

solve $F(x) = 0$ on $\mathbb{R}^n$

Consider another mainstream applied math problem:

- Given continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$, find $x \in \mathbb{R}^n$ so that

$$F(x) = 0.$$

- That is, solve n nonlinear equations in n unknowns.
- *No one has anything positive to say about this problem:*
  - no guarantee of existence
  - no guarantee of uniqueness
  - no effective theory of approximation

solve $F(x) = 0$ on compact, convex $K \subset \mathbb{R}^n$

But we can change the problem minimally, and have something positive to say:

- Assume $K \subset \mathbb{R}^n$ is compact and convex.
- Seek $x \in K$ so that $F(x) = 0$.
- Theorem. There is $x \in K$ so that

$$F(x) \cdot (y - x) \geq 0 \qquad \text{for all } y \in K$$

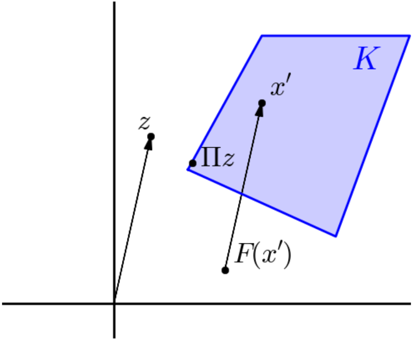
  which is a variational inequality

solve $F(x) = 0$ on compact, convex $K \subset \mathbb{R}^n$

Theorem. There is $x \in K$ so that

$$F(x) \cdot (y - x) \geq 0 \quad \text{for all } x \in K.$$

Proof. $x' \mapsto \Pi(x' - F(x'))$, as map on K, has a fixed point.



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2. dot-product for projection on a closed, convex $K \subset \mathcal{H}$:

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3. rewriting of "min f on convex $K \subset \mathbb{R}^n$":

$$u: \qquad \nabla f(u)\cdot(v-u) \ge 0 \qquad \text{for all } v \in K$$

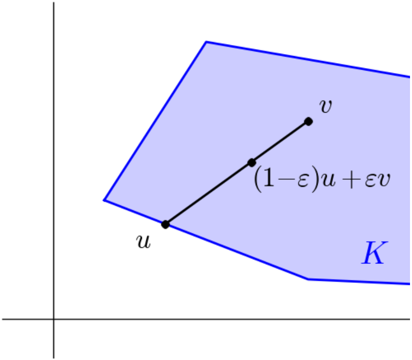
4. gets existence for *any* continuous nonlinear eqns on compact, convex $K \subset \mathbb{R}^n$:

$$x: \qquad F(x)\cdot(y-x) \ge 0 \qquad \text{for all } y \in K$$

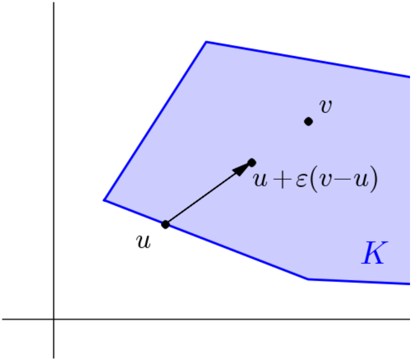
on convex K

if K is convex and $u, v \in K$ and $0 \le \varepsilon \le 1$ then

$$(1-\epsilon)u + \epsilon v = u + \epsilon(v-u) \in \mathcal{K}$$



$(1-\epsilon)u + \epsilon v$ as a linear combination in K



$u + \epsilon(v-u)$ as a vector from u directed into K

Outline

problems you can write as variational inequalities

obstacle problem example

three variational inequalities for glaciers

elastic membrane over obstacle

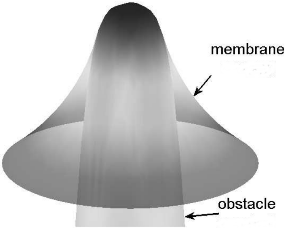
- elastic membrane $z = u(x,y)$ minimizes energy

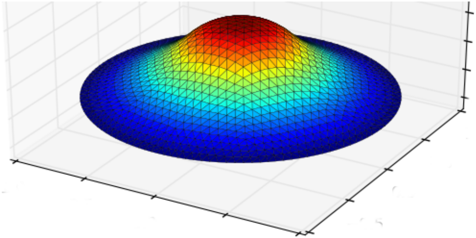
$$J[v] = \int_\Omega \frac{1}{2}|\nabla v|^2 - fv$$

where f is upward force on the membrane

- if surface $v(x,y)$ is above an obstacle $\psi(x,y)$ then it's in convex set

$$\mathcal{K} = \left\{v \in H_0^1(\Omega)\,:\, v \geq \psi\right\}$$





variational inequality for obstacle problem

- if $u \in \mathcal{K}$ is minimizer and if $v \in \mathcal{K}$ and if $0 \le \epsilon \le 1$ then

$$\begin{aligned} 0 &\le J[u + \epsilon(v-u)] - J[u] \\ &= \epsilon \int_\Omega \nabla u \cdot \nabla(v-u) - f(v-u) + \epsilon^2 \int_\Omega |\nabla(v-u)|^2 \end{aligned}$$

- thus as $\epsilon \to 0$, we know that $u \in \mathcal{K}$ satisfies

$$\int_\Omega \nabla u \cdot \nabla(v-u) - f(v-u) \ge 0 \qquad \forall v \in \mathcal{K}$$

  which is the variational inequality formulation
- also written:

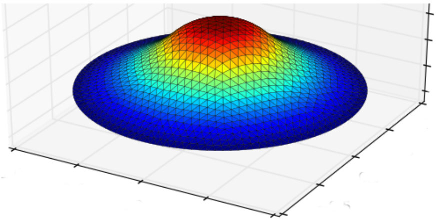
$$\langle \nabla J(u), v-u \rangle \ge 0 \qquad \forall v \in \mathcal{K}$$

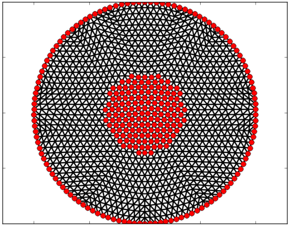
PDE for obstacle problem

- where $u > \psi$, the variational inequality implies $-\nabla^2 u = f$
  - the standard PDE for an *unobstructed* elastic membrane
  - $-\nabla^2 u = f$ is "Poisson equation"
- an engineer would say

  the membrane $u(x, y)$ solves $-\nabla^2 u = f$ except when it is in contact with the obstacle

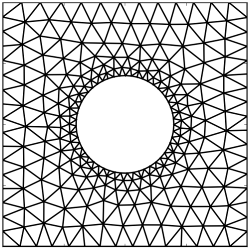
- but the set on which the contact happens is *a priori* unknown . . .





finite elements

- the finite element method (FEM) was built on variational *equalities*, i.e. “weak formulations”
- so variational inequalities play well with FEM
- FEM represents (approximates) function spaces on pretty meshes like this:



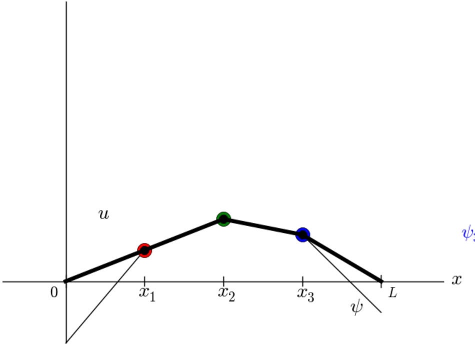
the 3-point, one-dimensional obstacle problem

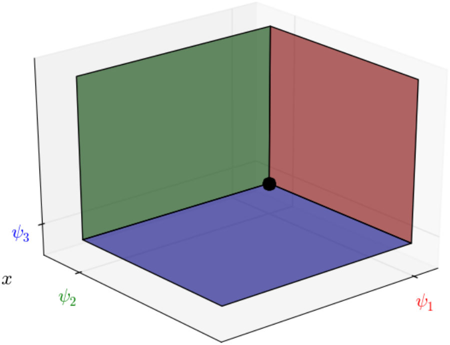
- for example, consider a one-dimensional obstacle problem
- with an equally-spaced 3-point mesh
- and a constant force $f(x) = f_0$
- so $\Omega = [0, L]$ has mesh points $\{x_1, x_2, x_3\}$
- the energy is just a quadratic function in $\mathbb{R}^3$:

$$\begin{aligned} J[v] &= \int_0^L \frac{1}{2}(v')^2 - f_0 v \\ &\approx \frac{1}{\Delta x}\left(v_1^2 + v_2^2 + v_3^2 - v_1 v_2 - v_2 v_3\right) - f_0 \Delta x (v_1 + v_2 + v_3) \end{aligned}$$

- and $\psi(x)$ and $u(x)$ represented by just three values each

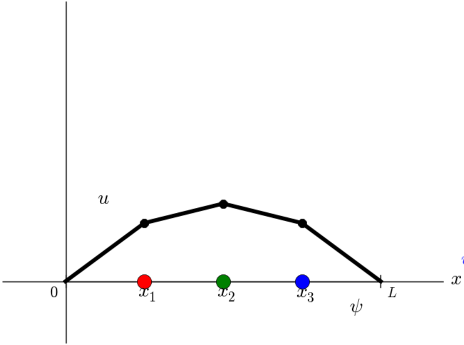
a 3-point case of the obstacle problem

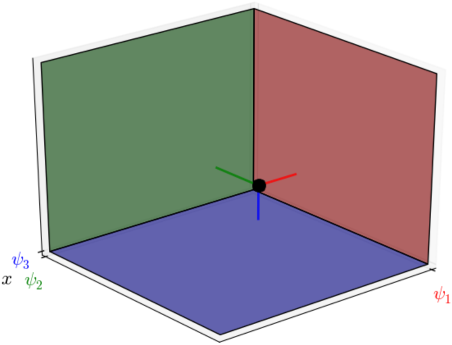




1: zero force $f_0 = 0$, one-hump obstacle

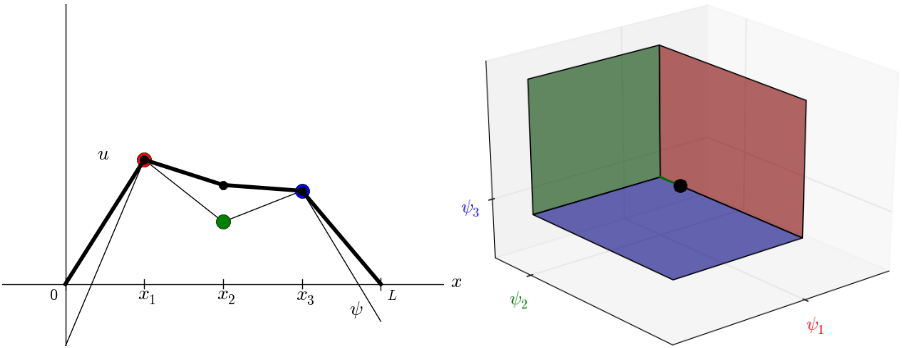
a 3-point case of the obstacle problem





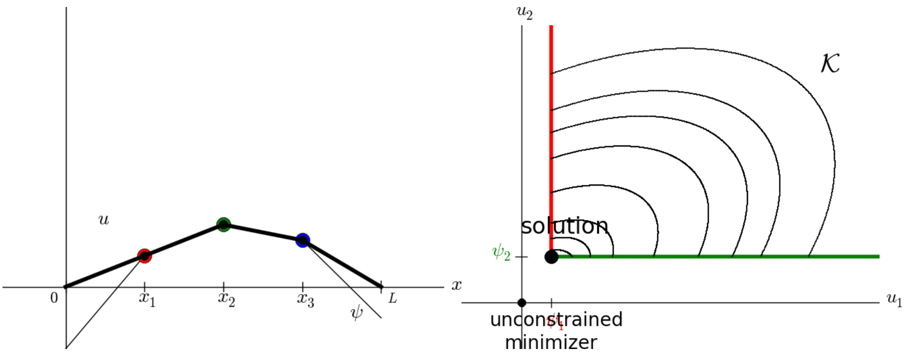
2: upward force $f_0 > 0$, flat obstacle

a 3-point case of the obstacle problem



3: downward force $f_0 < 0$, two-peak obstacle

a 3-point case of obstacle problem



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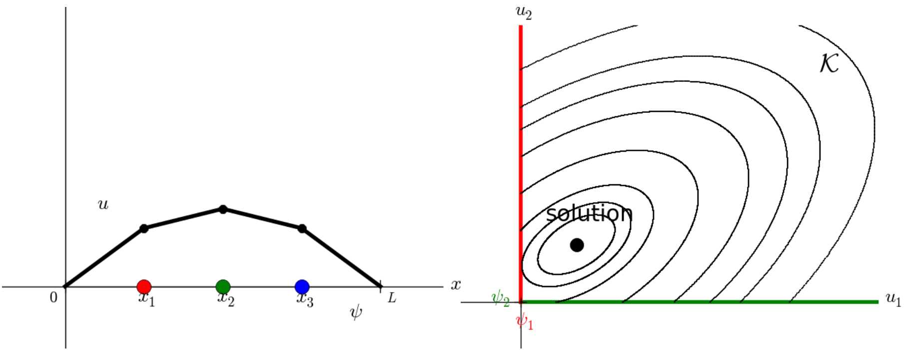

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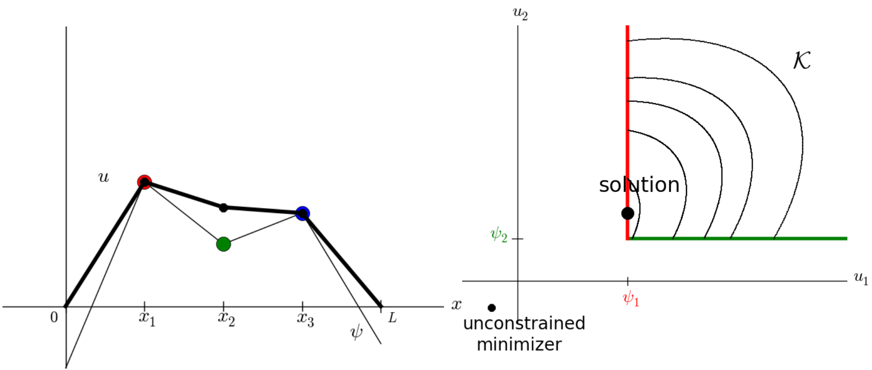

3: downward force $f_0 < 0$, two-peak obstacle

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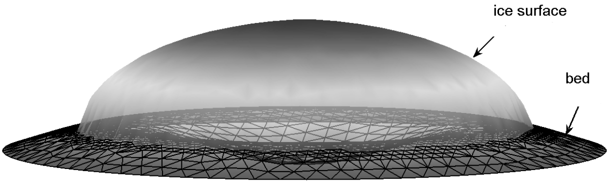
problems you can write as variational inequalities

obstacle problem example

three variational inequalities for glaciers

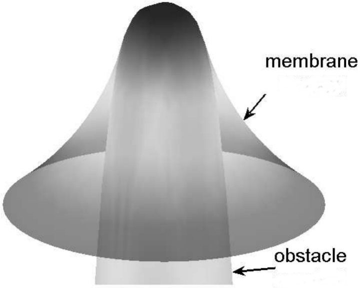
the steady-climate question for ice sheets

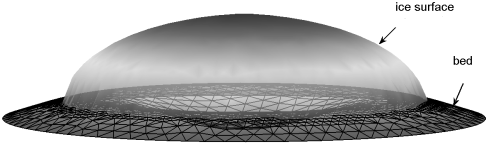
- suppose a steady-state climate
- where it snows some places and melts in others
- the ice flows into areas where there is melting
- questions:
  - what land is covered by ice sheets?
  - how thick are these sheets?



ice sheet geometry: an obstacle analogy

- ice surface $s(x, y)$
  $\sim$ *membrane*
- bedrock $b(x, y)$
  $\sim$ *obstacle*





v.i. 1: steady ice sheet surface "equation"

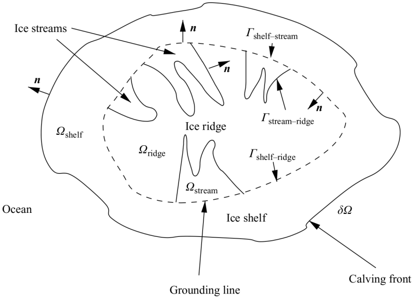
- ice sheet surface equation (so-called "SIA") applies only on domain where $s > b$
- let $h = s - b$, the ice sheet thickness
- equation applies only where $s > b \iff h > 0$
- define $p = n + 1$ where $n \approx 3$ for shear-thinning ice
- change variables $h = u^{(p-1)/(2p)}$
- define convex set $\mathcal{K} = \{v \in W_0^{1,p}(\Omega), v \geq 0\}$
- Theorem (Jouvet-Bueler 2012). There is $u \in \mathcal{K}$ solving the *steady transformed SIA*,

$$\int_\Omega \left(\mu|\nabla u - \Phi(u)|^{p-2}(\nabla u - \Phi(u))\right) \cdot \nabla(v - u) \geq \int_\Omega \alpha(u)(v - u)$$

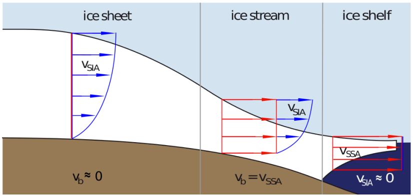
for all $v \in \mathcal{K}$

marine ice sheets: overview

- marine ice sheets are full of free boundaries:
  - boundary between floating ("shelf") and grounded
  - boundary between sliding ("stream") and not ("sheet")
  - boundary between wet base and dry base
- the Antarctic ice sheet is a marine ice sheet



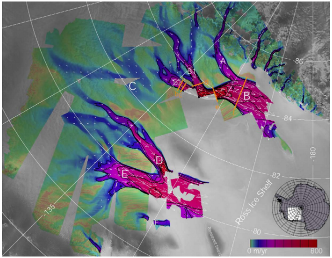
cartoon from (Schoof, 2006)

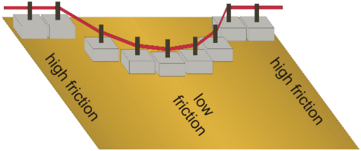


cartoon from (Martin et al., 2011)

ice stream sliding: an analogy

- ice stream is a viscous membrane with basal stresses
- ice streams emerge where basal resistance is sufficiently low
- a basal resistance model:
  - Coulomb friction, with
  - a yield stress distribution τ_c
- Schoof's slider analogy





v.i. 2: ice stream velocity "equation"

- let $q = 1 + \frac{1}{n}$ where $n \approx 3$ for shear-thinning ice
- $\mathbf{V}$ is ice stream velocity, $\mathbf{f} = -\rho g h \nabla s$ is driving stress, $\mathbf{F}$ is lateral stress along calving front
- Theorem (C. Schoof, 2006). There is unique velocity $\mathbf{U} = (u, v) \in W^{1,q}(\Omega)$ solving the *coulomb ice stream problem*. It minimizes

$$J[\mathbf{V}] = \int_\Omega \frac{2B}{q} h \, \|\mathbf{V}\|^q + \tau_c |\mathbf{V}| - \mathbf{f} \cdot \mathbf{V} - \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{V}$$

  with no constraint
- but $J[\mathbf{V}]$ is not smooth because of "$\tau_c|\mathbf{V}|$"

v.i. 2: ice stream velocity "equation"

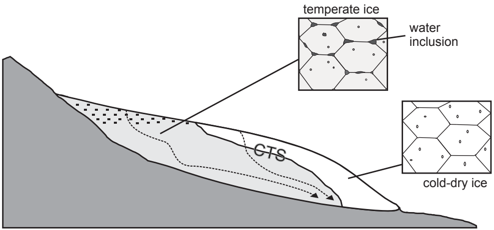
- Schoof started with a PDE for ice stream velocity (MacAyeal, 1989)
- then derived the variational inequality form: $\mathbf{U} \in W^{1,q}(\Omega)$ solves

$$\int_\Omega T_{ij}(\mathbf{U}) D_{ij} (\mathbf{V} - \mathbf{U}) + \tau_c (|\mathbf{V}| - |\mathbf{U}|) - \mathbf{f} \cdot (\mathbf{V} - \mathbf{U}) \geq \int_{\partial\Omega} \mathbf{F} \cdot (\mathbf{V} - \mathbf{U})$$

  for all $\mathbf{V} \in W^{1,q}(\Omega)$
- and then got $J[\mathbf{V}]$ on previous slide

some glaciers have cold ice

- to a glaciologist, ice is "cold" or "temperate"
  - cold ice has temperature below $0°$C
  - temperate ice is at $0°$C, but with liquid water
- temperature u, flow velocity $\mathbf{V}$
- heat flux is $\mathbf{q} = -k\nabla u + \rho c \mathbf{V} u$
  - conductive flux nearly zero in temperate ice ($\nabla u \approx 0$)
- viscous dissipation causes heating at rate S



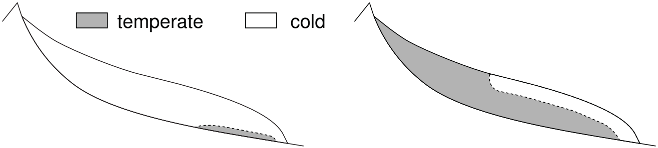
[figures from A. Aschwanden]

v.i. 3: cold-ice temperature in polythermal glacier

- in cold ice, everyone knows the steady temperature solves

$$0 = \nabla \cdot (k\nabla u - \rho c \mathbf{V} u) + S$$

- but where is the free boundary of the cold ice? (the "CTS")



- define convex set
  $\mathcal{K} = \{\phi \in H^1(\Omega) \,\big|\, \phi \le 0, \phi$ satisfies b.c.s$\}$
- Theorem (Gillispie-Bueler, in prep). there exists $u \in \mathcal{K}$ s.t.

$$\int_\Omega (k\nabla u - \rho c \mathbf{V}) \cdot \nabla(\phi - u) \ge \int_\Omega S(\phi - u)$$

for all $\phi \in \mathcal{K}$

variational inequalities for ice: a summary

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  - 3 for the cold ice is *not* a minimization

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- and that's it on variational inequalities for today