Conservation in free-boundary fluid layer models

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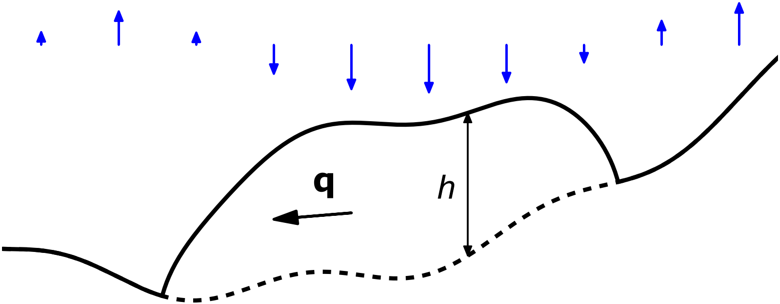
Outline

1. The problem I'm worried about:
   Time-stepping free-boundary fluid layer models.

2. Practical consequences:
   Limitations in reporting discrete conservation.
   Numerical weak free boundary solution needed.

A fluid layer in a climate



- mass conservation for a layer:

$$h_t + \nabla \cdot \mathbf{q} = f$$

  - h is a thickness: $h \geq 0$
  - mass conservation PDE applies *only where* $h > 0$
  - $\mathbf{q}$ is flow (vertically-integrated)
  - source f is "climate"; $f > 0$ shown downward

Examples

glaciers

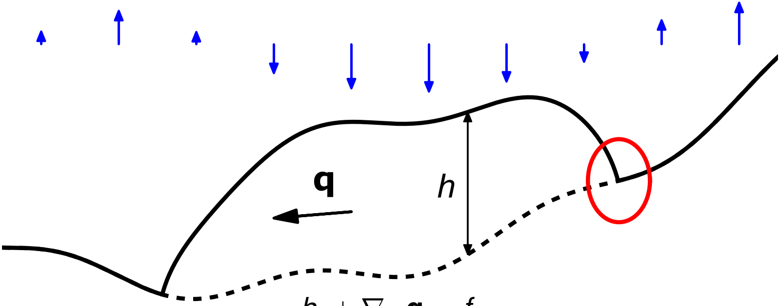
ice shelves & sea ice

tidewater marsh

tsunami inundation

and subglacial hydrology, supraglacial runoff, surface hydrology, . . .

A fluid layer in a climate: *the troubles*



$$h_t + \nabla \cdot \mathbf{q} = f$$

- $h = 0$ *and what else* at free boundary?
  - shape at free boundary depends on both $\mathbf{q}$ and f
- $f < 0$ not "detected" by model where $h = 0$
  - how to do mass conservation accounting?
- $f \approx 0$ threshold behavior
  - $h > 0$ as soon as $f < 0$ switches to $f > 0$

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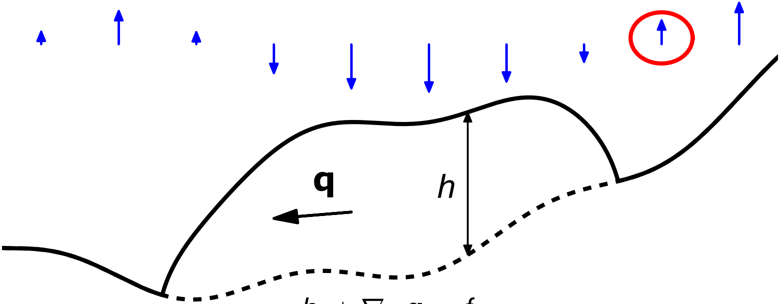

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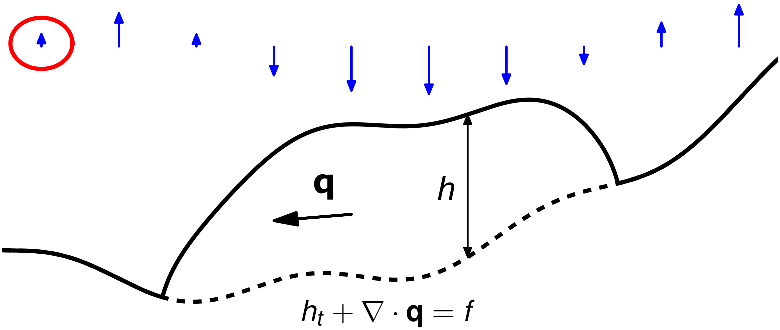

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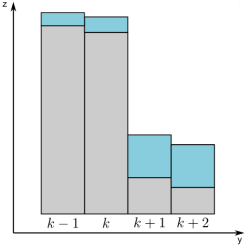
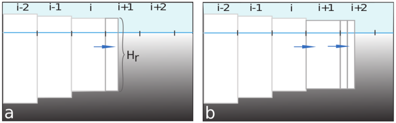
Anyone faced these problems before?

- yes, of course!
  - generic result: *ad hoc* schemes for finite volume/difference mass conservation near the free boundary



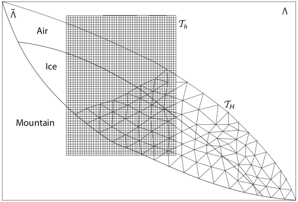
glacier ice
on steep terrain

(Jarosch, Schoof, Anslow, 2013)



volume-of-fluid method at ice shelf fronts

(Albrecht et al, 2011)



volume-of-fluid method (on fine grid) at glacier surface

(Jouvet et al 2008)

Anyone faced these problems before?

- yes, of course!
  - generic result: *ad hoc* schemes for finite volume/difference mass conservation near the free boundary
- I don't mind "`if ...then ...`" in my code, *but* I want to know what mathematical problem it reflects!
- my goals:
  - redefine the problem so free boundary is part of solution
  - use numerical schemes which automate the details

Numerical models *must* discretize time

$$h_t + \nabla \cdot \mathbf{q} = f \qquad \to \qquad \frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

- semi-discretize in time: $H_n(x) \approx h(t_n, x)$
- the new equation is a "single time-step problem"
  - a PDE in space where $H_n > 0$
  - this PDE is the "strong form"
- details of flux $\mathbf{Q}_n$ and source F_n come from time-stepping scheme
  - forward/backward Euler, trapezoid, RK all o.k.
  - note low regularity of $h(t, x)$ for x near margin

Weak form incorporates constraint

- define:

$$\mathcal{K} = \left\{ v \in W^{1,p}(\Omega) \,\middle|\, v \geq 0 \right\} = \text{admissible thicknesses}$$

- define: $H_n \in \mathcal{K}$ solves the weak single time-step problem if

$$\int_\Omega H_n(v-H_n) - \Delta t\, \mathbf{Q}_n \cdot \nabla (v-H_n) \geq \int_\Omega (H_{n-1} + \Delta t\, F_n)\,(v-H_n)$$

  for all $v \in \mathcal{K}$

  - derive this *variational inequality* from:
    - the strong form *and*
    - integration-by-parts *and*
    - arguments about $H_n = 0$ areas

Theorem: weak solves strong

Theorem. Assume $\mathbf{Q}_n = 0$ when $H_n = 0$. Assume $H_n \in \mathcal{K}$ solves weak single time-step problem and is smooth. Then

1. "interior condition" on set where $H_n > 0$:

$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

2. on set where $H_n = 0$:

$$H_{n-1} + \Delta t\, F_n \leq 0$$

re part 2:

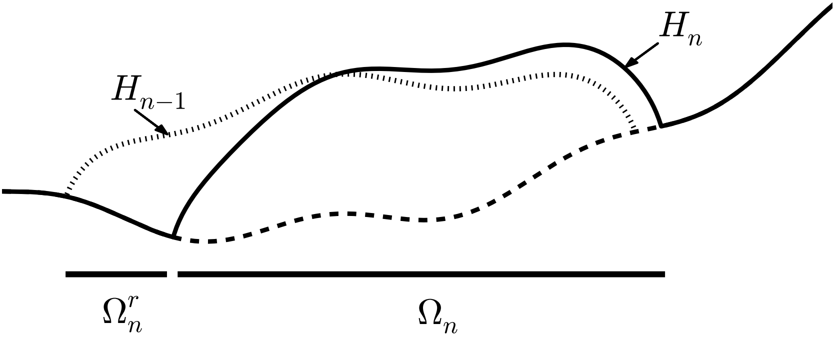
- "climate is negative enough to remove old thickness"
- assumption "$\mathbf{Q}_n = 0$ when $H_n = 0$" is needed
  - ...yes, we are talking about a *layer*

Subsets for reporting conservation

- suppose H_n solves the weak single time-step problem
- define

$$\Omega_n = \operatorname{supp} H_n = \{H_n(x) > 0\}$$

$$\Omega_n^r = \left\{ \ H_n(x) = 0 \text{ and } H_{n-1}(x) > 0 \ \right\} \quad \leftarrow \text{retreat set}$$



Reporting discrete conservation

- define:

$$M_n = \int_\Omega H_n(x)\,dx \qquad \textit{mass at time } t_n$$

- then

$$\boxed{\Delta t\,(-\nabla\cdot\mathbf{Q}_n + F_n)}$$

$$\begin{aligned} M_n - M_{n-1} &= \int_{\Omega_n} \boxed{H_n - H_{n-1}}\,dx + \int_{\Omega_n^r} 0 - H_{n-1}\,dx \\ &= \Delta t \left(0 + \int_{\Omega_n} F_n\,dx\right) - \int_{\Omega_n^r} H_{n-1}\,dx \end{aligned}$$

- new term:

$$R_n = \int_{\Omega_n^r} H_{n-1}\,dx \qquad \textit{retreat loss during step } n$$

Reporting discrete conservation: *claim*

- we want to "balance the books" for the model user
- the retreat loss R_n is not balanced by the climate
  - yes, R_n is caused by the climate, but we don't know what *computable integral* it balances
- a numerical model must track three time series:
  - mass at time t_n: $M_n = \int_\Omega H_n(x)\,dx$
  - climate over fluid-covered region:

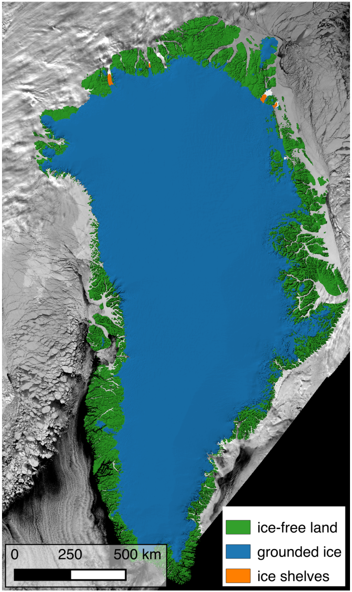
$$C_n = \Delta t \int_{\Omega_n} F_n\,dx \approx \int_{t_{n-1}}^{t_n} \int_{\Omega_n} f(t,x)\,dx\,dt$$

  - retreat loss: $R_n = \int_{\Omega_n^r} H_{n-1}\,dx$
- now it is balanced:

$$M_n = M_{n-1} + C_n - R_n$$

I am driven by practical modeling

- practical ice sheet modeling (e.g. Greenland at right)
- icy region nearly-fractal and disconnected
- currently in PISM*:
  - explicit time-stepping
  - free boundary by truncation
- want for PISM:
  - implicit time steps *with free boundary*
  - better conservation accounting to user



* = Parallel Ice Sheet Model, `pism-docs.org`

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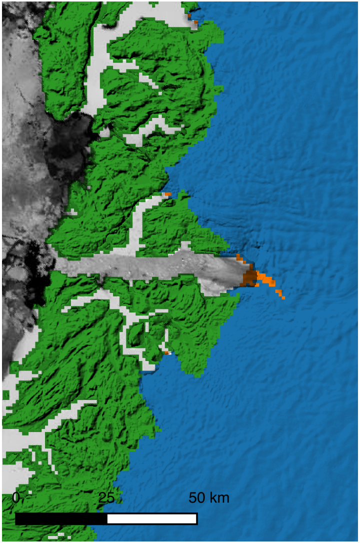

PISM whole Greenland at 900m
see Aschwanden C51A-0245 on 12/19

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Numerical solution of the weak problem

the weak single time-step problem:

- is nonlinear because of constraint (even for $\mathbf{Q}_n$ linear in H_n)
- can be solved by Newton method modified for constraint
  - reduced set method
  - semismooth method
- scalable implementations of both in PETSc 3.5
  - SNESVI class

Well-posedness of the weak problem

- I've been agnostic on form of $\mathbf{Q}_n$
  - except "$\mathbf{Q}_n = 0$ where $H_n = 0$" (i.e. it's a layer)
- but form of $\mathbf{Q}_n$ matters
  - for well-posedness and for numerical solutions
- cases to study:

$\mathbf{Q}_n = \mathbf{X}(x) H_n$	*transported layer*
$\mathbf{Q}_n = -k \nabla H_n$	*linear diffusion*
$\mathbf{Q}_n = -\nabla(H_n^\gamma) = -\gamma H_n^{\gamma-1} \nabla H_n$	*porous medium*
$\mathbf{Q}_n = -H_n^\alpha \lvert\nabla(H_n + b)\rvert^\beta \nabla(H_n + b)$	*shallow ice approx.* *& diffusive shallow water*
$\mathbf{Q}_n =$ *worse (non-local)*	*ice shelf flow & sea ice & ...*

- variational inequality is generally monotone
  - generally coercive if $\mathbf{Q}_n \sim -\nabla H_n$

Two numerical examples

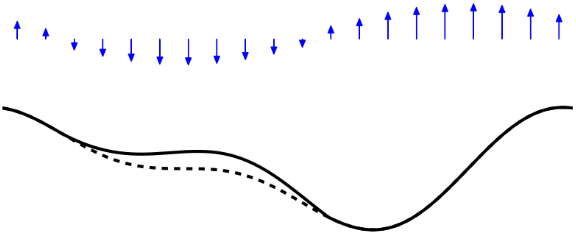
pop quiz:

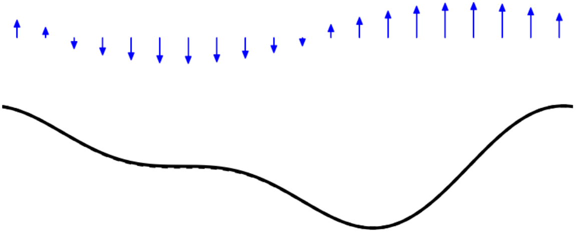
- same equation

$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = f$$

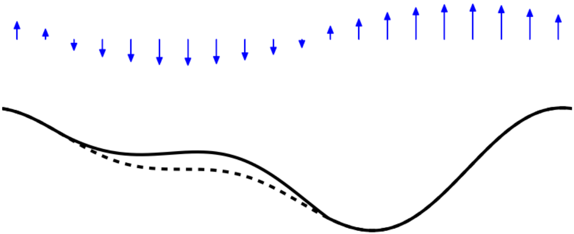
- same climate f
- same bed shape
- same constrained-Newton scheme

how different are the $\mathbf{Q}_n$?

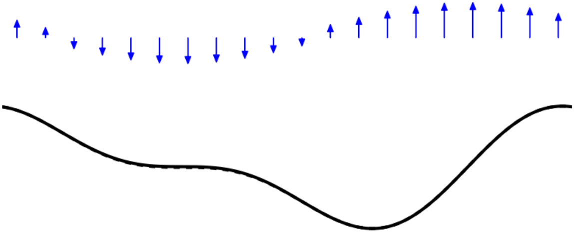




Two numerical examples



$\mathbf{Q}_n = v_0 H_n$
hyperbolic (constant vel.)



$\mathbf{Q}_n = -\Gamma |H_n|^{n+2}$
$\cdot |\nabla h_n|^{n-1} \nabla h_n$
highly-nonlinear diffusion

Terminal slide

- I'm considering layer flow problems:
  - model has conservation eqn: $h_t + \nabla \cdot \mathbf{q} = f$
- suggestions:
  - *include* constraint on thickness: $h \geq 0$
  - pose single time-step problem *weakly* as variational inequality
  - solve single time-step problem numerically by constrained-Newton method
- claim: for *any* numerical approach,
  - exact discrete conservation requires tracking *retreat loss*
    - in addition to computable integrals of climate
  - but it *isn't really possible* except in $\Delta t \to 0$ limit