Introduction to the mathematics of ice flow

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6 NOVEMBER, 2004 UPDATE: HALFAR, ETC.

Based on joint work with Craig Lingle[1], Jed Brown[2], Dave Covey[1], and Latrice Bowman[2].

Outline

1. VERY BRIEF derivation of eqns for ice-as-a-fluid

2. the cold, shallow ice approximation = two "PDE"s

3. isothermal, cold, shallow ice eqn = single PDE

4. a solution

5. another solution

6. deeper view of the shallow ice flow "PDE"

7. numerical verification for nasty coupled nonlinear PDEs

Ice is a fluid

So what does "ice is a fluid" mean mathematically?

From Fowler, *Mathematical Models in the Applied Sciences*,

> In describing the motion of a fluid (either liquid or gas, and in fact, even in some circumstances, solid) it is assumed that the locally averaged velocity vector field $\mathbf{u}$ is a twice continuously differentiable function of $\mathbf{x}$ and t. This is the continuum hypothesis. Equations describing the motion of a fluid are those of conservation of mass and momentum, together with an equation of state. In addition, there is an energy equation for the temperature T, but in most cases it uncouples and is of no concern.

NOTE: COUNT EQNS!.

Ice is a "non-Newtonian" fluid

Above equation count assumes linear or "Newtonian" *rheology*, i.e. a linear relation between applied force (stress) and deformation (strain) for small hunk of fluid. **Not so for ice.** We add a "constitutive relation":

$$(\text{strain}) \sim (\text{stress})^n \qquad \longleftarrow \text{"Glen ice flow law"}$$

Experiment suggests (Glen 1955, Goldsby-Kohlstedt 2001)

$$1.8 \leq n \leq 4$$

but n varies throughout ice.

n is actually stress- and temperature- dependent!

NOTE: Draw hunk picture.

Equations of a non-Newt. fluid

Suppose: ρ is density, $\mathbf{u}$ is velocity, p is pressure, σ is the stress tensor, $\dot{\epsilon}$ is the strain rate tensor, τ is "deviatoric stress tensor" ($\tau_{ij} = p\delta_{ij} + \sigma_{ij}$), and T is temperature.

conserve mass	$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$
conserve momentum	$\rho\left[\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = \nabla \cdot \sigma - \rho \mathbf{g}$
equation of state	$\rho = f(p, T)$
conserve energy	$\rho c_p\left(T_t + \mathbf{u} \cdot \nabla T\right) = k\nabla^2 T + \tau_{ij}\dot{\epsilon}_{ij}$
constitutive relation	$\dot{\epsilon}_{ij} = A(T)\|\tau\|^{n-1}\tau_{ij}$

This is a "cold" fluid if we are talking about ice: only one (solid) phase.

NOTE: COUNT EQNS.

NOTE: Equations are more nonlinear than Navier-Stokes!.

Ice = incomp. slow non-Newt. fluid

- Assume incompressibility, that is, constant density ρ.

- Assume temp. part of constitutive relation is "Arrhenius":

$$A(T) = A_0 \exp\left(\frac{-Q}{RT}\right).$$

- Assume accelerations are small ("Stoke's flow"). That is, remove inertial terms from conservation of momentum:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} \qquad \rightsquigarrow \qquad 0.$$

(Also stated: Reynolds number equals zero.)

Eqns of ice as a fluid

conserve mass $\qquad\qquad\qquad \nabla \cdot \mathbf{u} = 0$

conserve momentum $\qquad\quad \nabla p = \nabla \cdot \tau - \rho \mathbf{g}$

constitutive relation $\qquad\; \dot{\epsilon}_{ij} = A_0 \exp\left(\frac{-Q}{RT}\right) \|\tau\|^{n-1} \tau_{ij},$

conserve energy $\qquad\quad\; \rho c_p \left(T_t + \mathbf{u} \cdot \nabla T\right) = k \nabla^2 T + \tau_{ij} \dot{\epsilon}_{ij}$

[defn of strain rate tensor] $\; \dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

where *unknown fields* are: velocity $\mathbf{u}$, pressure p, deviatoric stress tensor τ, strain rate tensor $\dot{\epsilon}$, and temperature T.
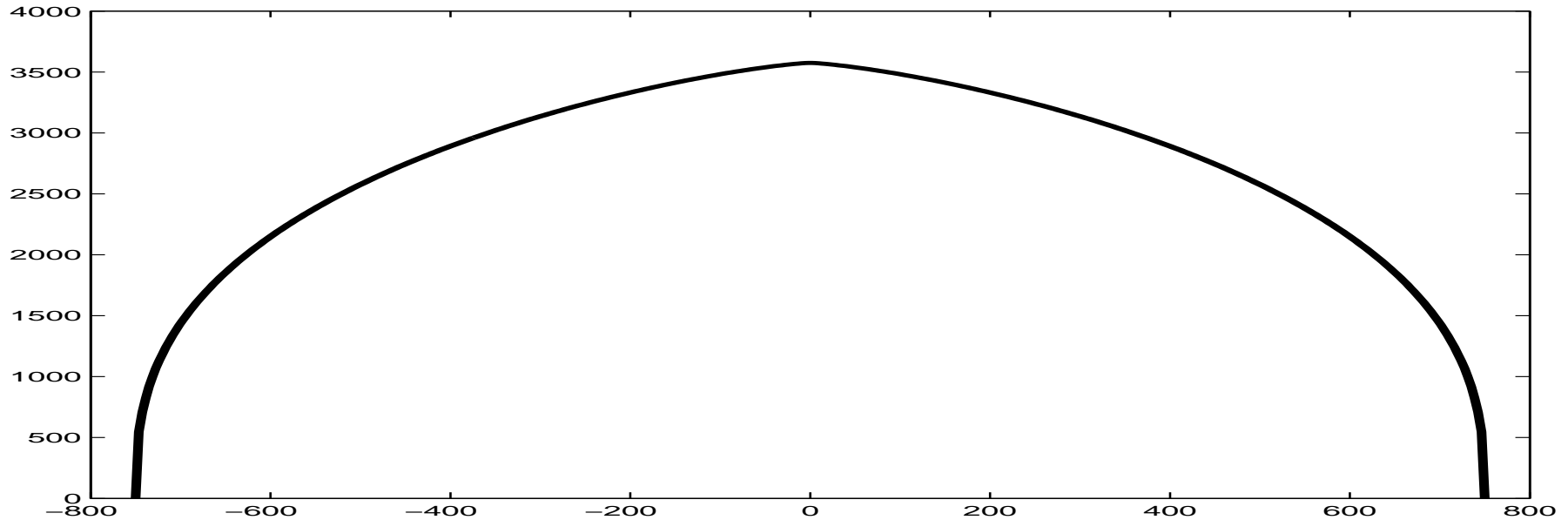
NOTE: Well posed initial value problems? Who knows?.
NOTE: Equation of state not needed because incompressible..
NOTE: General constitutive relation: $\dot{\epsilon}_{ij} = F(\tau, T, p)\tau_{ij}$. Example: Goldsby-Kohlstedt..
NOTE: What are these crazy "deviatoric stress" and "strain rate tensors"? Look in
"solid mechanics" or "metallurgy"..

But an ice sheet is shallow!

A standard test sheet. Yes, but what units?

Reality check: East Antarctic ice sheet has

- area 9.86×10^6 km^2; thus width at least 1700 <u>km</u>
- mean ice depth 2630 <u>m</u>

Shallow ice approximation

Let

$$\epsilon = \frac{d}{L}$$

where

- d is typical ice sheet depth and
- L is typical ice sheet width.

Scale the ice fluid equations appropriately. Drop terms which are of order ϵ, ϵ^2.

NOTE: Reference for technique: Fowler, *Mathematical Models . . . .*

Shallow ice approximation, cont.

Conclude:

1. pressure is proportional to depth,

2. τ_{xz}, τ_{yz} are only remaining (deviatoric) stresses,

3. effective shear stress (i.e. norm of τ_{ij}) is proportional to depth times surface slope,

4. *horizontal* heat conduction part of conservation of energy can be ignored.

Shallow means *a lot* fewer eqns!

Full incompressible, slow non-Newtonian fluid model for ice is eleven (11) scalar differential equations.

Shallow ice approximation reduces this to TWO scalar evolution equations, coupled by a (vertical) integral.

Simplifying assumptions for talk

For the rest of the talk:

- Ice frozen to bed, that is, no basal sliding.

- $n = 3$.

- One horizontal dimension. (So x and z remain but no y. Velocity $\mathbf{u} = (u, w)$—just x, z components.)

Eqns of shallow ice approximation

$$h_t = a - \left(\int_0^h u \, dz \right)_x, \qquad h \geq 0, \qquad \text{(FLOW)}$$

$$T_t + uT_x + wT_z = KT_{zz} + cA(T)\|\tau\|^4, \qquad \text{(TEMP)}$$

with coupling relations

$$u = -2h_x \int_0^z A(T)\|\tau\|^2 \rho g(h - \zeta) \, d\zeta, \qquad w = -\int_0^z u_x \, d\zeta$$

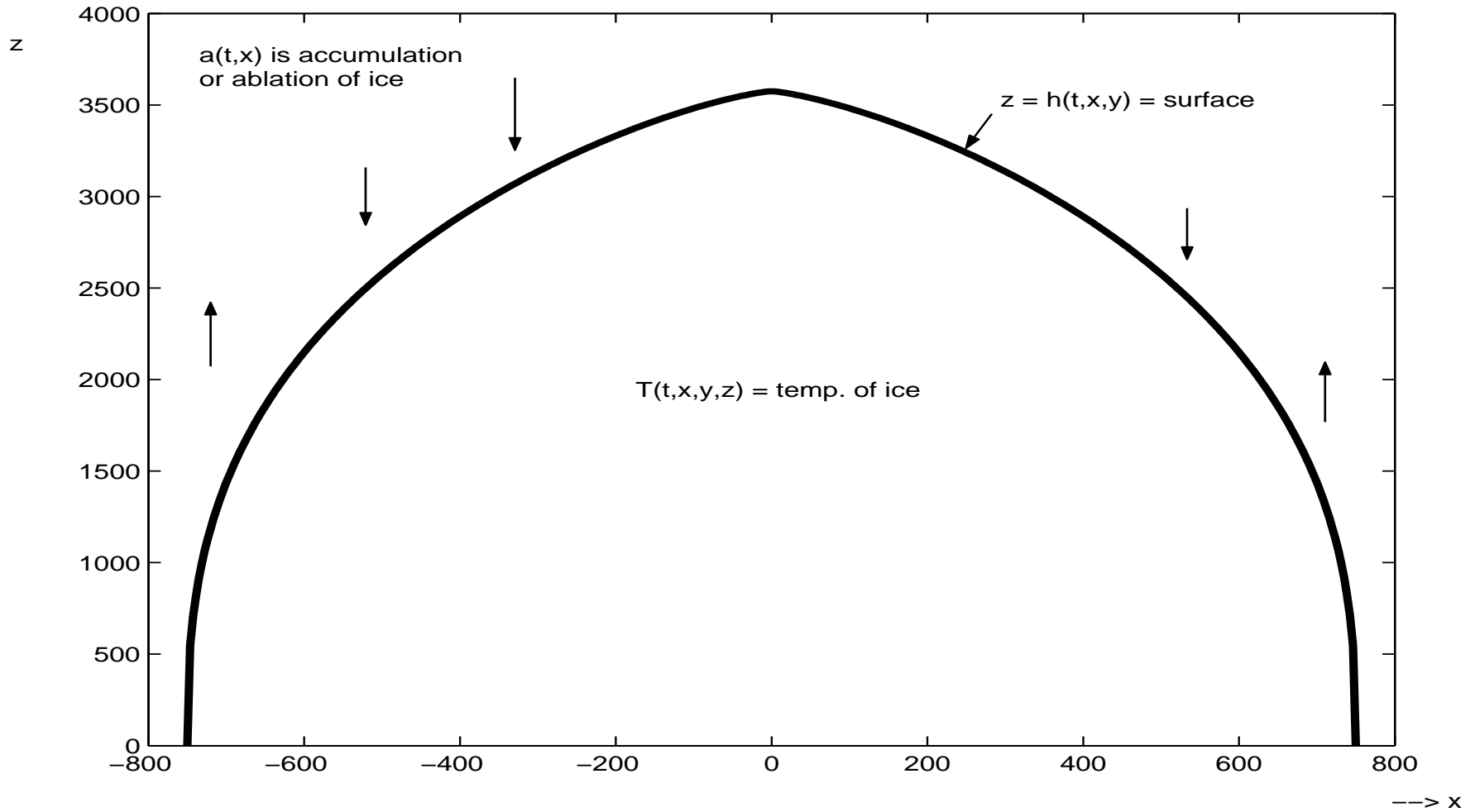Also $\qquad \|\tau\| = \rho g(h - z)|h_x|, \qquad A(T) = A_0 \exp\left(\frac{-Q}{RT} \right).$

To find: $h = h(t, x)$, $T = T(t, x, z)$.

NOTE: BCs for (TEMP): T(surface) given; $T_z\big|_{\text{base}}$ =(geothermal heat flux).
NOTE: Subscripts $(\ )_t$, $(\ )_x$, $(\ )_z$ are partial derivatives.

Too much chicken scratching!

O.K. It's still a big mess of equations. But here's a picture:

Free boundaries

Note that $h(t, x)$ solves the (FLOW) equation as well as a boundary condition at a *free boundary*, that is, the moving margin.

Note that $T(t, x, z)$ solves the (TEMP) equation within the ice but that the upper boundary is moving in time according to (FLOW). Also (TEMP) is coupled to (FLOW). Thus T also solves a *free boundary problem*.

An open problem

Consider the coupled (FLOW) and (TEMP) equations of the shallow ice approximation.

Open problem: Show that an initial and (free) boundary value problem is *well-posed*.

Well-posedness

Definition. An initial/boundary value problem for a differential equation (or system thereof) is *well-posed* if

(a) the problem in fact has a solution;

(b) this solution is unique; and

(c) the solution depends continuously on the data given in the problem.

How hard might it be to prove these for coupled (FLOW) and (TEMP)? Uncoupled?

Shallow ice approximation again

How to uncouple?:

$$h_t = a - \left(\int_0^h u \, dz \right)_x, \qquad h \geq 0, \qquad \text{(FLOW)}$$

$$T_t + u T_x + w T_z = K T_{zz} + cA(T) \|\tau\|^4, \qquad \text{(TEMP)}$$

along with

$$u = -2h_x \int_0^z A(T) \|\tau\|^2 \rho g (h - \zeta) \, d\zeta, \qquad w = - \int_0^z u_x \, d\zeta,$$

$$\|\tau\| = \rho g (h - z) |h_x|, \qquad A(T) = A_0 \exp\left(\frac{-Q}{RT} \right).$$

Isothermal Cold Shallow Ice Eqn (*ICSIE*)

Assume temperature T is constant. Then we can integrate vertically for u in (FLOW). Then integrate again. Velocities u, w can be determined from $h = h(t,x)$. Get:

$$h_t = a + \left(\frac{\Gamma}{5} h^5 |h_x|^2 h_x \right)_x \qquad \text{(ICSIE)}$$

where

- $h = h(t,x) \geq 0$ thickness (also surface height),

- $a = a(t,x)$ accumulation/ablation,

- $\Gamma > 0$ physical constant

It's a *diffusion*.

Heat eqn, the classical diffusion

$$u_t = (K u_x)_x \qquad\qquad \text{(HEAT)}$$

Typically the *conductivity* or *diffusivity* $K > 0$ is constant.

Comparison:

$$
\begin{array}{ccc}
\text{(HEAT)} & \text{vs.} & \text{(ICSIE)} \\
u(t, x) & \longleftrightarrow & h(t, x) \geq 0 \\
K > 0 & \longleftrightarrow & \frac{\Gamma}{5} h^5 |h_x|^2 \geq 0
\end{array}
$$

(ICSIE) as a "diffusion"

Think

$$h_t = a + (Dh_x)_x \qquad \text{(ICSIE)}$$

for "diffusivity" D and "source" a, where

$$D = \frac{\Gamma}{5} h^5 |h_x|^2.$$

But $D = 0$ at some locations! Where? **NOTE: DRAW PICTURE**.

It's a *nonlinear* and *singular* diffusion. Can one prove it is well-posed? Can one solve it?
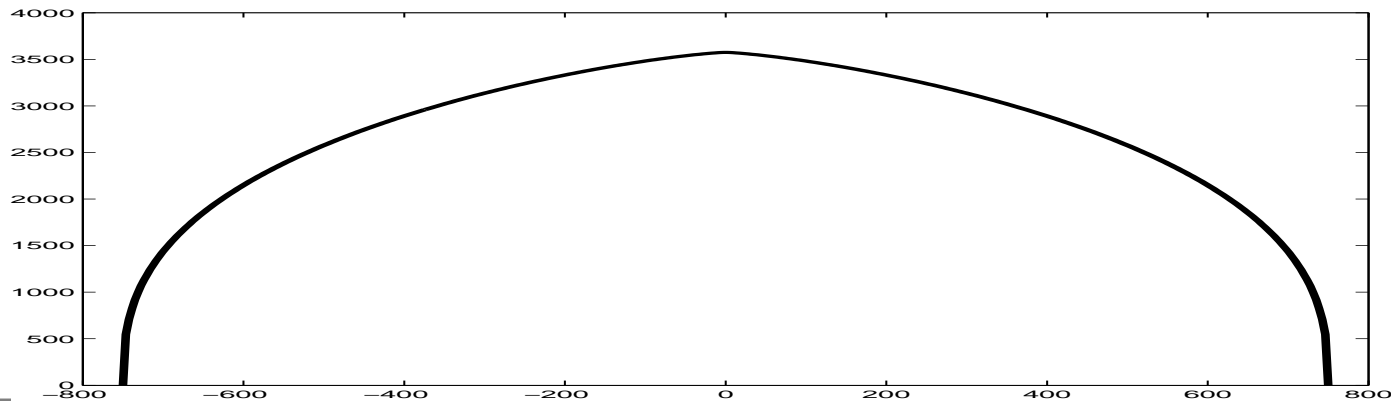
Exact steady state solutions

Suppose $a = a(x)$. Suppose we want to solve the steady state ($h_t = 0$) version of ICSIE: $0 = a + (Dh_x)_x$.
This is two step process (Nye, 1950s):

1. integrate a: $Dh_x = -\int^x a(\xi)\,d\xi$;

2. since $Dh_x = \pm(\Gamma/5)h^5(h_x)^3$, we can take $1/3$ root and solve separable ODE (*as long as we can do this second integral*).

Get, for instance,

Exact time-dependent solutions

(Halfar 1981): There are solutions to (ICSIE) of the form

$$h(t, x) = \frac{1}{t^\alpha} v \left(\frac{x}{t^\beta} \right) ;$$

"similarity" solutions. Halfar finds α, β if $a \equiv 0$. $v = v(x)$ has compact support. In radial case, $\alpha = \frac{1}{9}$, $\beta = \frac{1}{18}$ (if $n = 3$).

Contribution of Bueler, Lingle, Kallen-Brown, Covey, Bowman (2004): Further similarity solutions with $a \neq 0$; in fact, a is proportional to h and can be positive or negative.

Halfar's time-dependent similarity solution
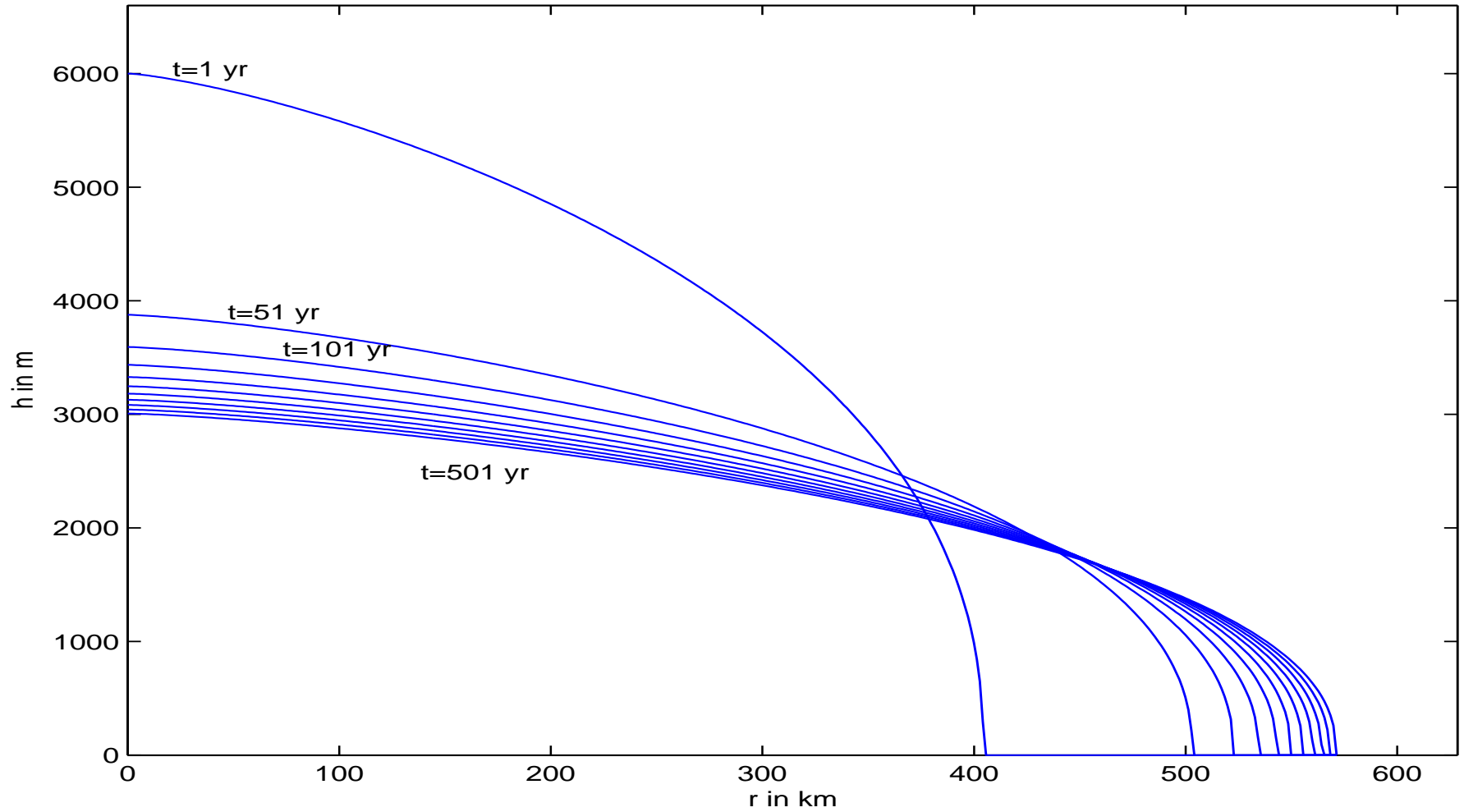
The conceptual difficulty

In (ICSIE), $h(t, x) \geq 0$. It does not appear in the PDE and must be enforced as a side condition.

How to do this correctly? Effect at boundary?

What *is* the correct dynamic of moving boundary in an ablation zone?

Well-posedness proven for (ICSIE)

December 2002:

> N. Calvo, J. I. Díaz, J. Durany, E. Schiavi, and
> C. Vázquez, *On a doubly nonlinear parabolic
> obstacle problem modelling ice sheet dynamics*,
> SIAM J. Appl. Math. **63** (2) 2002, 683–707

From "variational inequality" or "weak" formulation, they prove there *exists a unique solution* for $t > 0$ to (ICSIE) if a and $u(t = 0, x) = u_0(x)$ are bounded.

Furthermore, solutions which have similar initial data and similar accumulation remain close together. (Also, solutions bounded for any finite time.)

That is, (ICSIE) is well-posed.

How to prove it?

Transform: [$n = 3$ case]

$$u = h^{8/3},$$

to get

$$(u^{3/8})_t = a + \mu \left(|u_x|^2 u_x \right)_x. \qquad \text{(ICSIE')}$$

u is a *more regular quantity* than h.

Expression $Lu = \left(|u_x|^2 u_x \right)_x$ is "p-Laplacian" for $p = 3$; somewhat understood.

NOTE: Let x_m be position of margin. Suppose $a < 0$ is continuous near margin (generic case). In steady–state, $h \sim (x_m - x)^{1/2}$, so h' is singular. New quantity $u \sim (x_m - x)^{4/3}$ so u' not singular at margin..

Analog of next step

Calvo, et al, formulate (ICSIE') as an "*obstacle problem*."

Example. Consider a taut elastic membrane $z = u(x, y)$ over an obstacle $z = b(x, y)$. We know

$$u \geq b.$$

From a weak formulation (i.e. minimize energy of the membrane): if $u > b$ then

$$u_{xx} + u_{yy} = 0. \tag{MEMB}$$

NOTE: Draw membrane over obstacle..

Main point of obstacle/weak formulation

The location of the boundary "on which we apply the boundary condition $u = b$" is not known in advance. Determining the location is part of the "free boundary value problem."

Weak formulation allows inequality constraint to determine space of allowed functions, rather than requiring imposition of a boundary condition at an unknown-in-advance location.

Well-posedness for (ICSIE')

The condition $h \geq 0$ in (ICSIE), or $u = h^{8/3} \geq 0$ in (ICSIE'), is an "obstacle". That is, the surface of the ice does not extend below ground.

Method for proving existence (part of well-posedness) in steady state:

1. space of appropriately regular functions satisfying the constraint $u \geq 0$ (i.e. $h \geq 0$) is *closed* and *convex*

2. (highly nonlinear) functional $I[u] = \int \frac{1}{4}\mu|u_x|^4 + au$ is *coercive* and *convex* in u_x

3. general facts imply functional $I[u]$ has minimum on this space; minimizers satisfies $0 = a + \mu \left(|u_x|^2 u_x\right)_x$

Is Calvo, et al relevant in practice?

Yes.

A weak formulation of an obstacle problem is the first step of a *finite element method* (FEM). Wherein one solves inequalities, unfortunately . . . (!) Resulting FEM has clear boundary condition and some error bounds.

(Calvo et al did it only in one horizontal variable, where the boundary is easy to follow . . .)

Back to shallow ice approximation

Can one find exact solution?:

$$h_t = a - \left(\int_0^h u \, dz \right)_x, \qquad h \geq 0, \qquad \text{(FLOW)}$$

$$T_t + uT_x + wT_z = KT_{zz} + cA(T)\|\tau\|^4, \qquad \text{(TEMP)}$$

along with

$$u = -2h_x \int_0^z A(T)\|\tau\|^2 \rho g(h - \zeta) \, d\zeta, \qquad w = - \int_0^z u_x \, d\zeta,$$

$$\|\tau\| = \rho g(h - z)|h_x|, \qquad A(T) = A_0 \exp\left(\frac{-Q}{RT} \right).$$

Can one find exact solution?

YES! At least one can find solutions which are useful for numerical testing purposes.

Example. Solve

$$f(u, u_x, u_{xx}) = g(x) \qquad\qquad \text{(NASTY)}$$

where precise g doesn't matter.

Solution method. Choose $u(x)$ and call it your solution. Substitute u into the left side of (NASTY). Call the result g. You have now solved (NASTY) with some g. You are likely to be able to approximate your u numerically if (NASTY) is linearly stable.

Why this is useful for shallow ice

The shallow ice approximation has been the subject of many numerical simulations since ~ 1990. The methods used are relatively conservative numerical analysis (e.g. finite differences) and have been tested against steady state solutions.

But there has been no comparison to exact time-dependent solutions. Also no convergence evidence, rates.

Instead, "intercomparison of models" (e.g. EISMINT 1996, 2000) has been the main test method. When enough models agree and the results are scientifically credible, these results have been the basis of predictions and even policy.

Our current efforts

- verification of 3D thermocoupled shallow ice approximation numerical models by use of exact solutions;

- identification of numerical error causes (e.g. approximation of the friction heating term $\Sigma = A(T)\|\tau\|^4$);

- comparison of constitutive relations (e.g. G-K vs. Glen);

- inclusion of other effects: sliding, basal deformation, ice streams, ice sheets, etc.;

- a finite element version based on the weak inequality formulation;

- getting courage to try to prove well-posedness for thermocoupled models . . .

Last Slide: A Moral?

Sometimes one should not be intimidated by nonlinearities, complexities, and uncertainties in physical models.

In particular,

- exact solutions;
- abstract but clear formulations

may be achievable. And they help in practice (e.g. numerics)!