# *Ice Sheets and Obstacle Problems*

Ed Bueler

A mathematical work in progress,

part of ongoing ice sheet modeling, toward a complete Antarctica simulation,

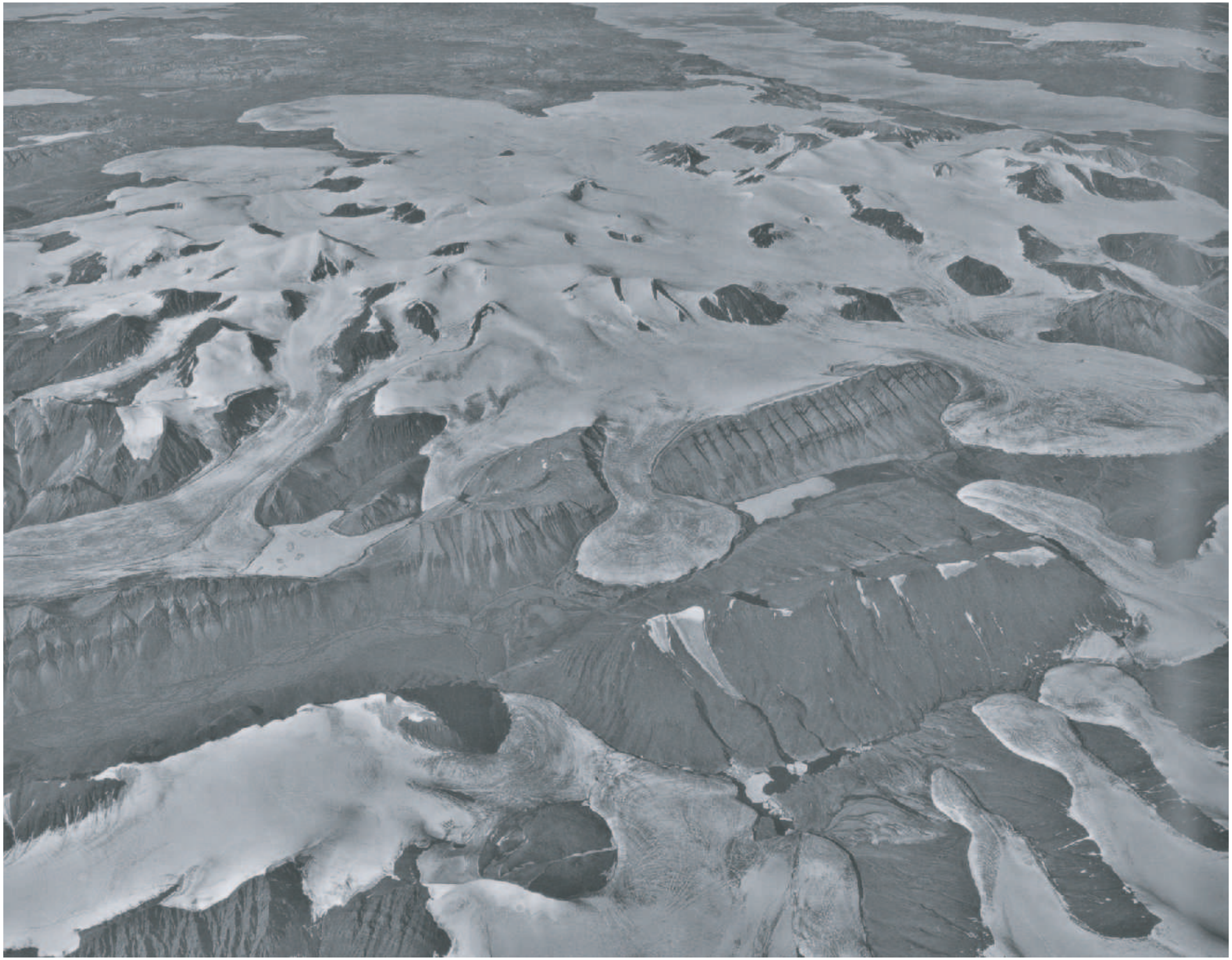
joint work with

Craig Lingle (Geophysical Institute),

Jed Kallen-Brown (graduate student, Mathematics),

and many others, on NASA grant NAG5-11371.

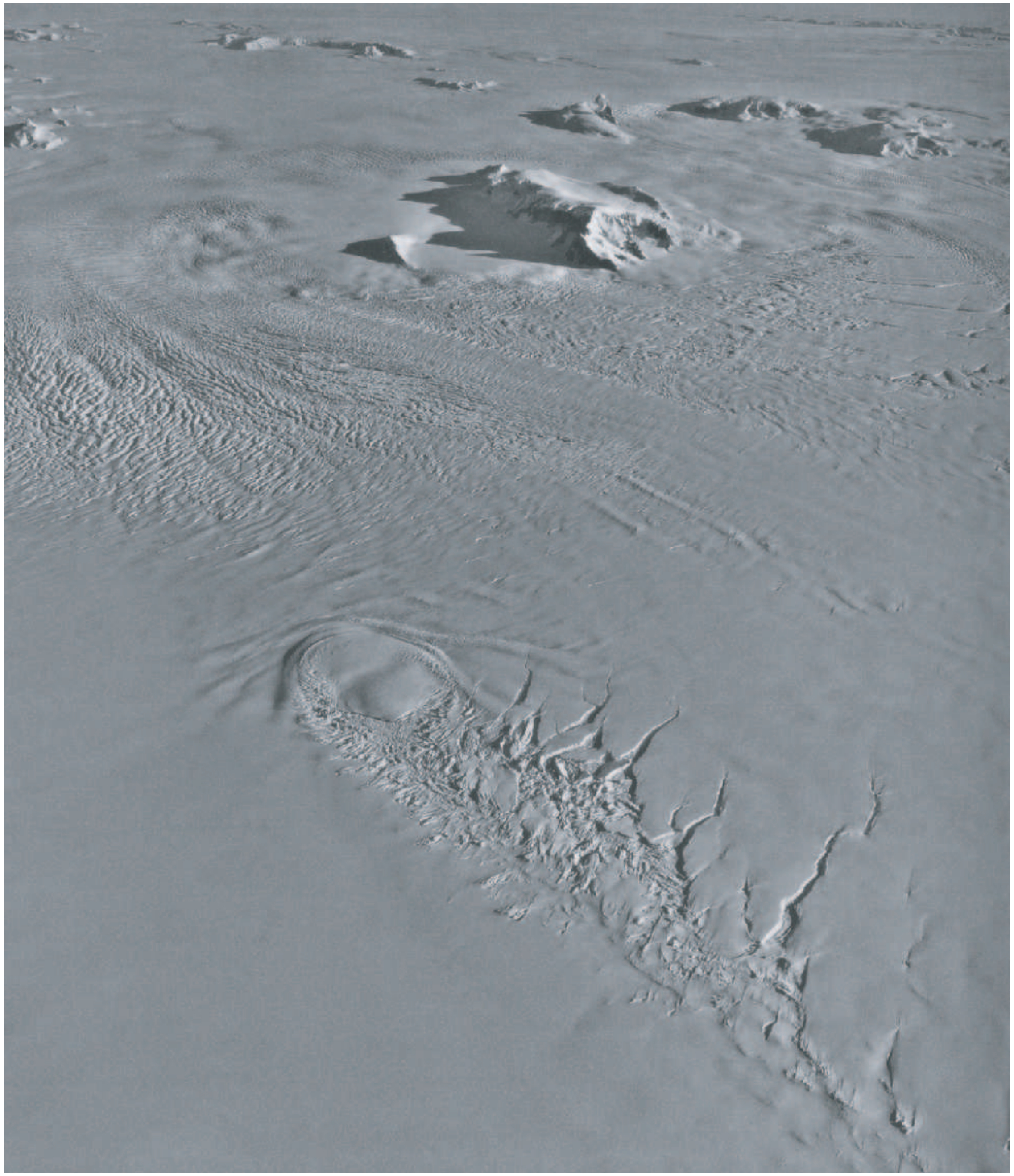
# *Ice flows. Like molasses.*



Small polar ice field on Axel Heiberg Island, Nunavut, Canada. *Photo 119, Post & LaChapelle 2000.*

Ed: *add at top, lose at bottom*

# *Really! It flows!*



Palmer Land, Antarctica. *Photo 131, Post & LaChapelle 2000.*

# *Shallow flow (mostly); steady geometry (nearly)*

“Polaris Glacier,” northwest Greenland. *Photo 122, Post & LaChapelle 2000.*

# *Shallow, grounded, “Glen-law” ice flow*

Denote:

- $h(t, x, y)$ is surface elevation—the *primary unknown*
- $b(t, x, y)$ is bed elevation, i.e. of the land underneath
- $a(t, x, y, z)$ is accumulation/ablation rate; at least yearly-averaged, in practice
- $\mathbf{U}_b$ is horizontal sliding at base [*more on this later!*]

Then we have the ISOTHERMAL SHALLOW ICE SHEET EQUATION:

$$h_t = a + \nabla \cdot \left[\Gamma(h-b)^{n+2}|\nabla h|^{n-1}\nabla h - (h-b)\mathbf{U}_b\right]$$

where $\Gamma > 0$ is constant—actually depends strongly on temperature—and

$$n \approx 3$$

SEEMS LIKE A PRETTY TOUGH NUT TO CRACK!

It is some kind of nonlinear diffusion-advection equation:

$$u_t = f + \nabla \cdot (D\nabla u) + \mathbf{X} \cdot \nabla u + c\,u$$

Ed: *column of ice*

# *Suppose "steady state"—note there is still flow!*

If we assume:

- no change over time in climate, so $a = a(x, y, z)$
- no change over time in bed elevation, so $b = b(x, y)$

then the solution has an equilibrium $h(x, y) = \lim_{t\to\infty} h(t, x, y)$.

$h(x, y)$ solves the *Steady* Isothermal Shallow Ice sheet Equation ("SISIE?"):

$$0 = a + \nabla \cdot \left[\Gamma (h - b)^{n+2} |\nabla h|^{n-1} \nabla h - (h - b)\mathbf{U}_b\right]$$

*Note*: the ice does flow in steady state. In fact, if velocity is $(u, v, w)$ and if $b(x, y) < z < h(x, y)$, then

$$(u(x, y, z), v(x, y, z)) = -\Gamma \left(\frac{n+2}{n+1}\right) \left((h - b)^{n+1} - (h - z)^{n+1}\right) |\nabla h|^{n-1} \nabla h + \mathbf{U}_b.$$

Vertical velocity $w(x, y, z)$ is given by incompressibility: $w_z = -u_x - v_y$.

Slogan: Geometry determines flow.

| Ed: *constant profile enclosing flow w non-flat bed* |
| --- |

# *Not yet a Mathematical Model: No BCs!*

Of interest—to me, anyway—is the *grounded* boundary, the “margin” of the ice sheet.

(Ice sheets on earth frequently flow into/onto the ocean. There they have a very different type of boundary!)

SO: What determines the location of a grounded margin?

ANSWER:

1. distribution (in 3D) of accumulation/ablation
2. bed slope
3. sliding at base

A well-posed version of SISIE must include these influences!

Ed: *add accum, ELA, sliding to prev*

## *Forget the flow within; where is the surface?*

PROBLEM:

Given $a(x,y,z)$, $b(x,y)$, and $\mathbf{U}_b$ [*more on this later!*], find $h(x,y)$ so that

1. $$0 = a + \nabla \cdot \left[\Gamma (h-b)^{n+2} |\nabla h|^{n-1} \nabla h - (h-b)\mathbf{U}_b\right]$$

2. $$h \geq b$$

There is a CHANGE OF PERSPECTIVE here:

- Glaciologist says SISIE (1 above) mainly tells us how geometry determines flow, because of the flux expression appearing in square brackets. The formula

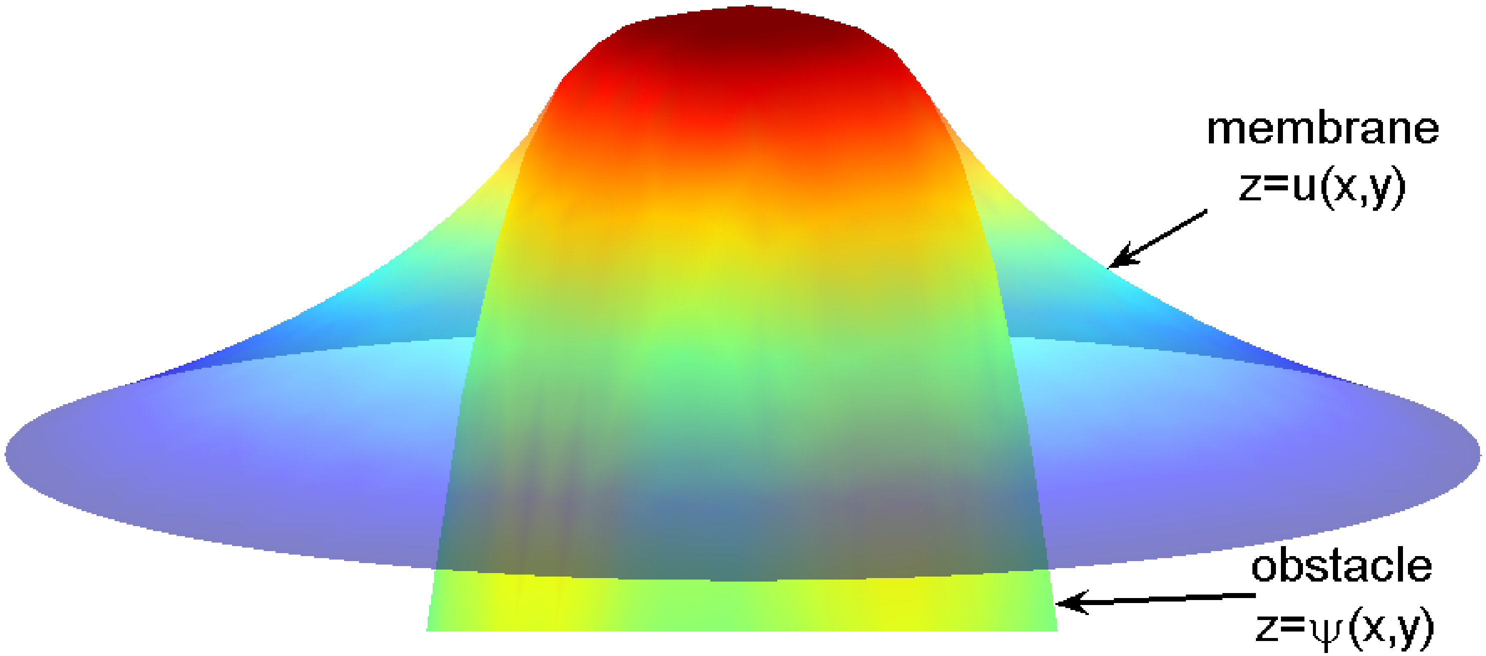
  $$\mathbf{Q} = -\Gamma (h-b)^{n+2} |\nabla h|^{n-1} \nabla h - (h-b)\mathbf{U}_b$$

  for (vertically-integrated horizontal) flux is evidently determined by geometry (thickness $h-b$ and surface gradient $\nabla h$). Glaciologist says "2 is obviously true–how can that help me?"

- Mathematician says SISIE and condition 2 above tell us how to determine the ice sheet geometry given the data $(a,b)$. The PROBLEM consisting of 1 and 2 is (morally) well-posed because 2 is (morally) a good boundary condition.

# *How can an inequality be a BC?*

EXAMPLE (the *classical obstacle problem*):



# *Classical obstacle problem, cont.*

We see for this obstacle problem that *stating a PDE is not enough*!

We do *have* a PDE: Where the membrane is *not* in contact with the obstacle,

$$-\nabla^2 u = f \qquad \text{(Poisson equation)}$$

where $f(x, y)$ is the force on the membrane. Note $f = 0$ in previous slide. (Notation: $\nabla^2 u = \nabla \cdot (\nabla u) = u_{xx} + u_{yy}$. The Poisson equation is a good model of a real elastic membrane when all gradients are modest.)

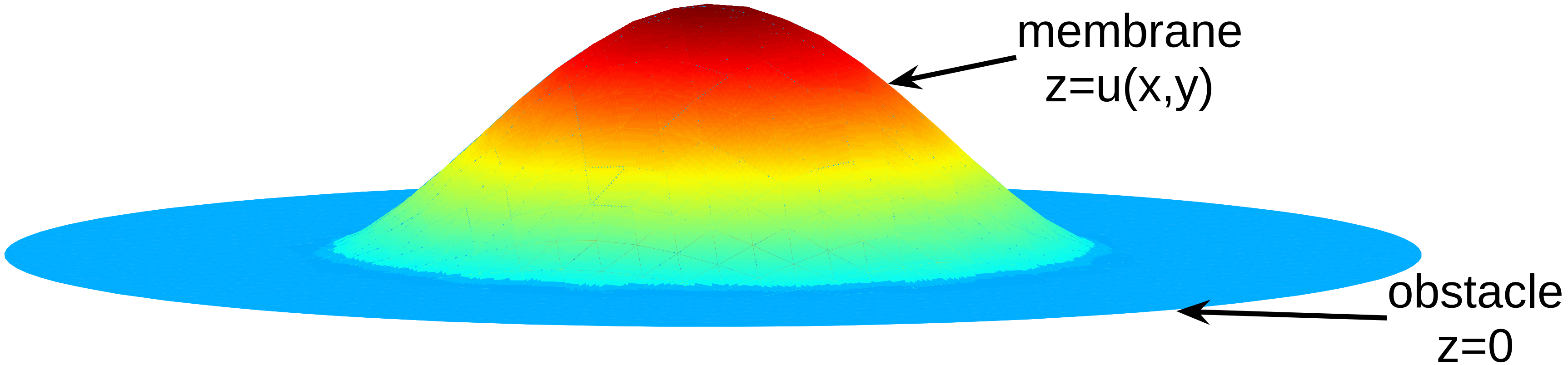
*Also* we have facts at the boundary of the region where the Poisson PDE applies: At the boundary of $R = \{(x, y) \,\big|\, u(x, y) > \psi(x, y)\}$ we have

- $u = \psi$, the position of the membrane is known, *and*
- $\nabla u = \nabla \psi$, the membrane is tangent to the obstacle and thus has known gradient.

But we don't know *where* the membrane touches the obstacle! (There is a free boundary.)

## *"That membrane thing don't look like an ice sheet to me!"*

Suppose $\psi = 0$ (the obstacle is flat) and $f > 0$ in center of membrane and $f < 0$ near boundary of the membrane:




This shows a solution to

$$0 = f + \nabla \cdot (1 \, \nabla u) \, ,$$

the Poisson equation, in the region where $u > \psi = 0$.

# *Classical obstacle problem as minimization*

The classical problem is a well-posed mathematical model only if we replace the PDE with a different mathematical construct.

Let $\Omega$ be the set of points $(x, y)$ where the membrane elevation is defined.
Let

$$\mathcal{K} = \{\text{functions } v(x,y) \text{ on } \Omega \text{ such that } v(x,y) \geq \psi(x,y)\}\,.$$

Define the *(potential) energy of the membrane*

$$J[v] = \int_\Omega \frac{1}{2}|\nabla v|^2 - fv$$

where $f(x, y)$ is the (distributed, vertical) force on the membrane.

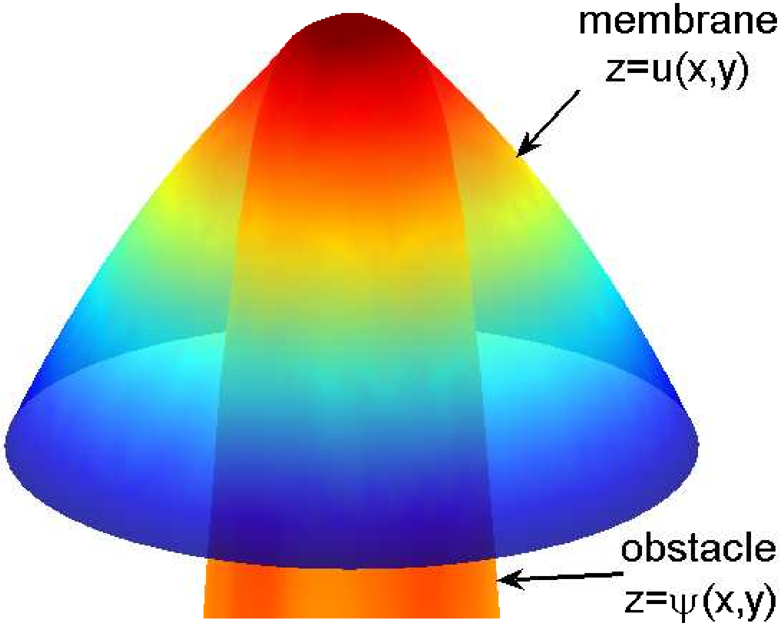
*Theorem*. If $\mathcal{K}$ is a space of appropriately smooth functions, the function $u(x, y)$ defined by the minimization

$$\min_{v \in \mathcal{K}} J[v]$$

exists, is unique, and depends upon $f$ and $\psi$ in a controlled way.

That is, the CLASSICAL OBSTACLE PROBLEM IS WELL-POSED.
One can solve the problem by finite elements; a short MATLAB code is available!

# *The membrane problem really is different from an ice sheet!*




In this picture the membrane has force $f(x, y)$ *which is positive everywhere* ($f \geq f_0 > 0$). Nonetheless the obstacle in contact with the membrane.

Ice is different:

$$a(x, y) > 0 \text{ implies } h(x, y) > b(x, y).$$

That is, *if you are in a place where it snows more than it melts each year, then there will be an ice sheet there!*

# *First bad* mathematical *news (for ice, and for the audience)*

We will *not* get away with thinking of the ice sheet surface as membrane which minimizes some (goofy) energy, constrained by the obstacle formerly known as the bed!

[Glaciologist: *Duh! The surface of an ice sheet is not a material surface! You don't just push it up or down, and the ice/snow on the surface one year becomes part of the flow below (which you seem to willfully ignore!) in the next year.*]

But there is a different, also "weak," formulation of the membrane problem which will have a (perfect!) analogy to the ice sheet problem:

*Theorem*. Recall $\mathcal{K} = \{v$ such that $v \geq \psi\}$. The following are equivalent:

$u$ solves $$\min_{v \in \mathcal{K}} J[v] = \int_\Omega (1/2)|\nabla v|^2 - fv$$

and

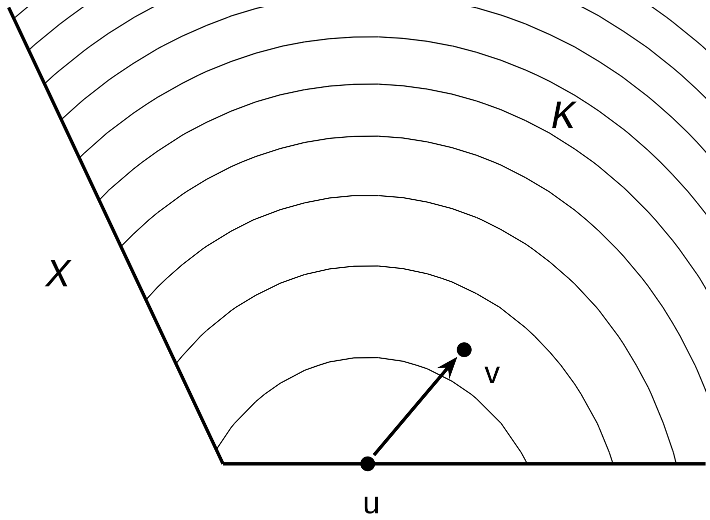
$u$ solves $$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u) \qquad \text{for all } v \in \mathcal{K}.$$

The latter is a *variational inequality*.

# *Variational inequality: a cartoon*

Let $\mathcal{X}$ be the set of *all* functions and let $\mathcal{K}$ be the set of admissible functions. (For the membrane: $\mathcal{K} = \{v$ such that $v \geq \psi\}$.)

Consider the "contour map" of $J[v]$. (For the membrane, $J$ is a quadratic function(al) which takes in a function $v$ and puts out the number $\int_\Omega (1/2)|\nabla v|^2 - fv$.)




Necessary properties of $\mathcal{K}$:

- closed
- convex

SLOGAN: $u \in \mathcal{K}$ minimizes $J[v]$ over $\mathcal{K}$ if all directions $v - u$ "heading into $\mathcal{K}$", that is, for $v \in \mathcal{K}$, make $J[v]$ increase because $J'[u](v-u) \geq 0$.

# “$J'[v](v-u)$” *means what?*

The variational inequality property of the minimizer $u$ is

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u) \qquad \text{for all } v \in \mathcal{K}$$

which can be rewritten

$$\int_\Omega \nabla u \cdot \nabla(v-u) - f(v-u) \geq 0 \qquad \text{for all } v \in \mathcal{K}.$$

That is, the derivative of $J$ appears:

$$J'[v]w = \int_\Omega \nabla u \cdot \nabla w - fw. \tag{1}$$

In fact,

$$J[v+w] = J[v] + J'[v]w + o(\|w\|)$$

can be proved, so (1) really does describe a (directional!) derivative [*in the infinite dimensional space of all functions $\mathcal{X}$*.]

# *Steady ice sheet as solution of obstacle problem*

Recall the PROBLEM: Given $a(x, y, z)$, $b(x, y)$, and $\mathbf{U}_b$, find $h(x, y)$ so that

1. $$0 = a + \nabla \cdot \left[\Gamma(h - b)^{n+2}|\nabla h|^{n-1}\nabla h - (h - b)\mathbf{U}_b\right]$$
2. $$h \geq b$$

Reformulate as an OBSTACLE PROBLEM: Given $a(x, y, z)$, $b(x, y)$, and $\mathbf{U}_b$, find $h(x, y)$ so that

$$\int_\Omega \left[\Gamma(h-b)^{n+2}|\nabla h|^{n-1}\nabla h-(h-b)U_b\right]\cdot\nabla(v-h) \geq \int_\Omega a(v-h) \qquad \text{for all } v \in \mathcal{K} = \{v \geq b\}.$$

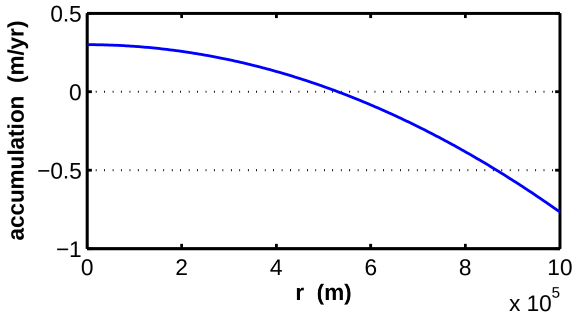
This version of the problem

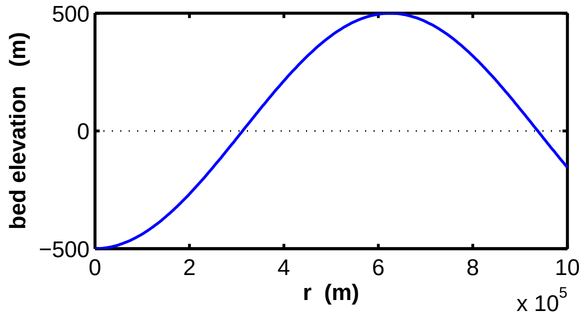
- is well-posed in some cases [*and I claim it is the right statement of the problem in all cases!*], and
- is *not* equivalent to the minimization of a functional unless the bed is flat

  Ed: *go back to cartoon*, and
- leads to a “one step to equilibrium” finite element (numerical) method [*a new thing!*].

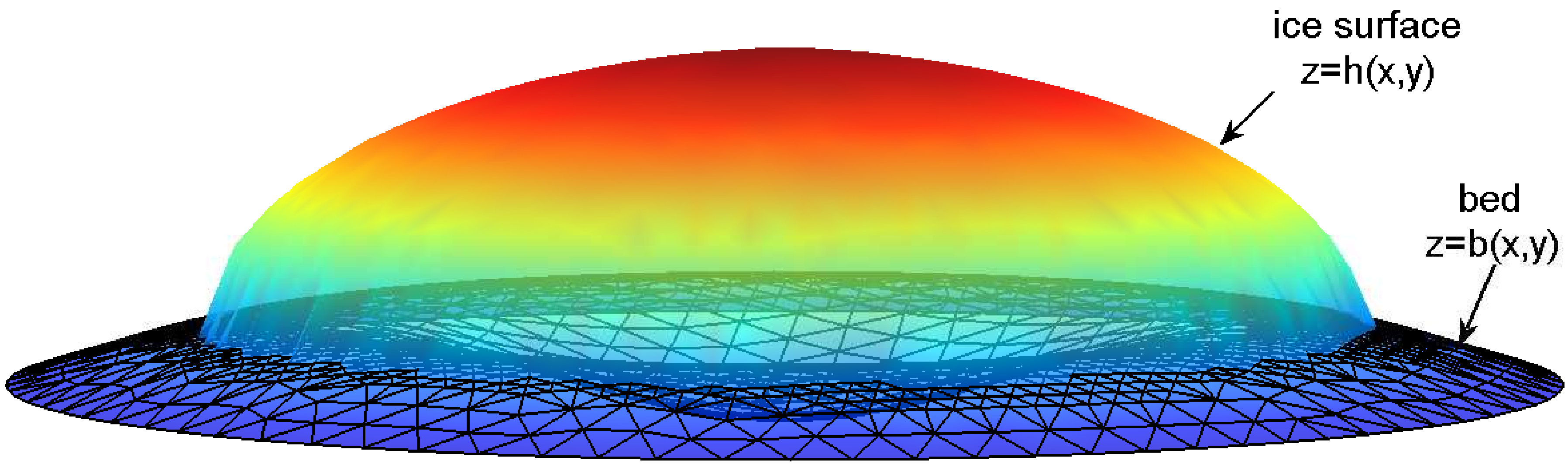
# *A result:*

Given the climate and the bed:

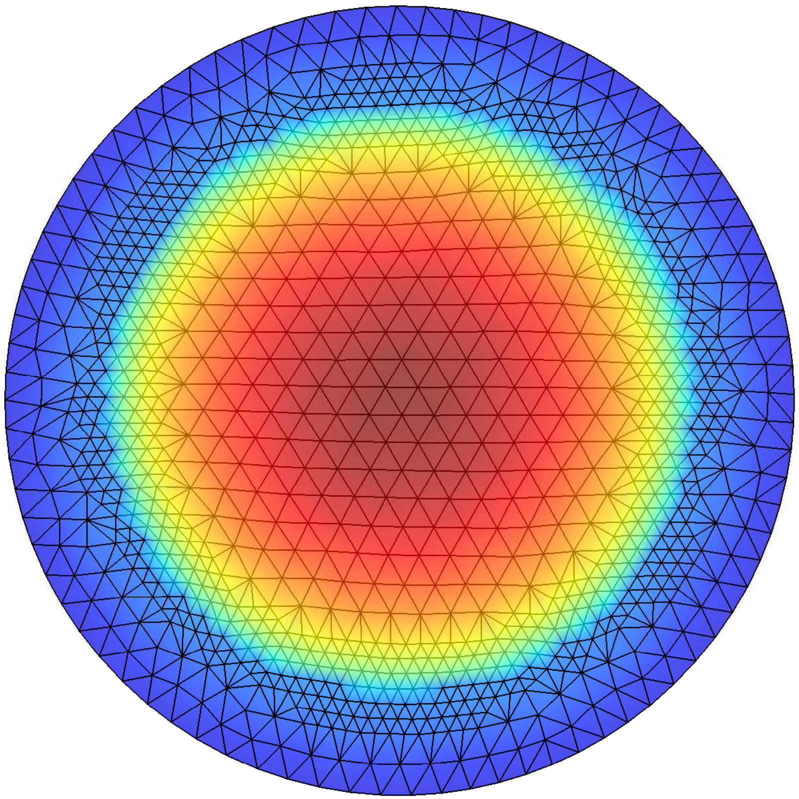


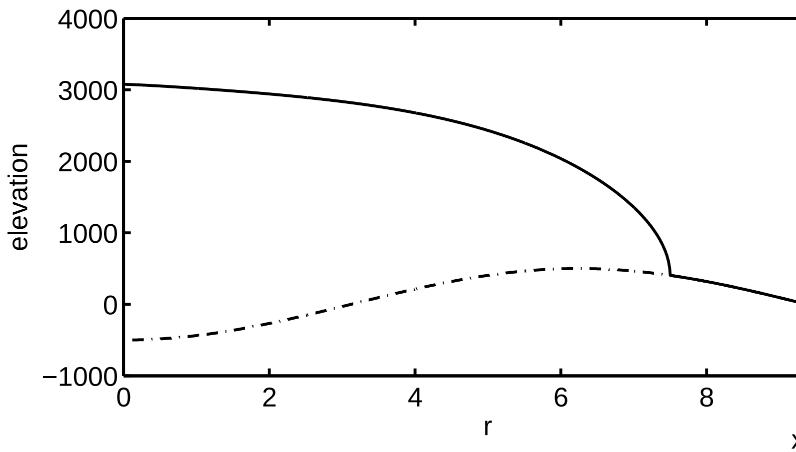




Get the ice sheet surface:



# *How? and How good?*





# *Prior work: Flat-bed, one dimension case.*

If we

- add time-dependence back into the problem

but we assume

- bed is flat $b = 0$
- problem has one spatial dimension
- accumulation depends only on horizontal position $a = a(t, x)$
- basal sliding velocity is a predetermined function $\mathbf{U}_b = \mathbf{U}_b(t, x)$

then we get the case addressed by Calvo, Díaz, Durany, Schiavi, Vázquez in 2002.

They were first to identify the ice sheet problem as a “obstacle problem.”

Interestingly, their results are not in the steady case but in the harder-to-formulate time-dependent case. They did *not*, apparently, observe that the steady problem for their case (when the bed is flat!) corresponds to minimization of a functional $J[h]$ in a space of admissible surfaces $\mathcal{K}$ or a variational inequality.

*Theorem*. A weak formulation of their problem is well-posed. Under certain circumstances, if the ice sheet is too large for the accumulation rate to support it, the grounded margin “waits” before retreating.

They also give a rather sophisticated “duality” algorithm for their problem.

# *Where do we stand mathematically?*

SISIE,

$$0 = a + \nabla \cdot \left[\Gamma(h-b)^{n+2}|\nabla h|^{n-1}\nabla h - (h-b)\mathbf{U}_b\right],$$

and the obvious fact that the ice surface must be at least as high as the bed ($h \geq b$) have together become . . .

a variational inequality

$$\int_\Omega \left[\Gamma(h-b)^{n+2}|\nabla h|^{n-1}\nabla h - (h-b)U_b\right] \cdot \nabla(v-h) \geq \int_\Omega a\,(v-h) \quad \text{for all } v \in \mathcal{K}$$

But this variational inequality is hard to understand because the margin occurs where the diffusivity

$$D = \Gamma(h-b)^{n+2}|\nabla h|^{n-1}$$

is singular (because thickness $h - b \to 0$ at the margin!).

$D$ also is zero where $|\nabla h| = 0$ is zero ("Raymond bumps"); the theory of the "$p$-Laplacian" $\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right)$ already handles this.

# *SISIE with transformed thickness*

We need a "trick," introduced by Calvo et al (2002) but done here in the non-flat bed case [for the first time, apparently]:

Consider the thickness $H = h - b$. We will rewrite the OBSTACLE PROBLEM in terms of a *power* of $H$

$$u = H^m = H^{(2n+2)/n}$$

The $u$-OBSTACLE PROBLEM: Given $a(x, y, z)$, $b(x, y)$, and $\mathbf{U}_b$, find $u = H^m$ so that

$$\int_\Omega \left[\tilde\Gamma |\nabla u - \Phi_b(u)|^{p-2}(\nabla u - \Phi_b(u)) - u^{1/m} U_b\right] \cdot \nabla(w - u) \geq \int_\Omega a(w - u) \quad \text{for all } w \in \mathcal{K}_0$$

where

$$\Phi_b(u) = -m\, u^{(n+2)/(2n+2)}\, \nabla b \quad \text{and} \quad \mathcal{K}_0 = \left\{ w \in W_0^{1,p}(\Omega) \,\middle|\, w \geq 0 \right\}.$$

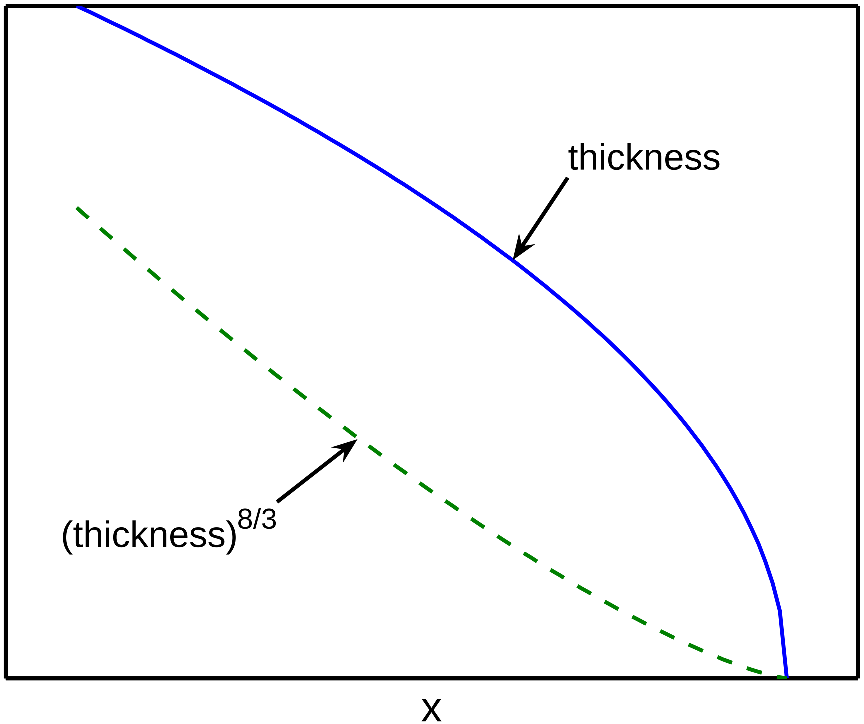
Here $p = n + 1 \approx 4$ and $\tilde\Gamma > 0$ is a multiple of $\Gamma$. Note $h = b + u^{1/m}$.

[*Note that the constraint on thickness* $H$*, or on* $u = H^m$*, is just* $H \geq 0$ *(*$u \geq 0$*, resp.)*.]

[*Note that the transformed* $u$*-problem* correctly *depends only on* $\nabla b$ *and not on* $b$ *itself!*]

# *Margin versus transformed margin*

When $n = 3$ then $m = 8/3$:




$H$ has infinite gradient at the margin

$u$ is actually tangent to the obstacle ($\psi = 0$)

# *One needs some assumptions to do math . . .*

To address the $u$-OBSTACLE PROBLEM we assume

- the bed is not too irregular ($b(x, y) \in W^{1,p}(\Omega)$),
- $a(x, y, z)$ is nonincreasing in $z$ (*in which case, for glaciological relevance, we might as well just assume* $a = a(x, y)$), is bounded above, and it is not too irregular, and
- $\mathbf{U}_b = \mathbf{U}_b(x, y, \sigma)$ is nondecreasing in the basal effective shear stress $\sigma = \rho g u^{1/m} \nabla(u^{1/m} + b)$, and it is not too irregular; it is Lipshitz in $\sigma$

[*Note that the form "$\mathbf{U}_b = \mathbf{U}_b(x, y, \sigma)$" assumes we have a "sliding law" at the base, which is a dubious model for a poorly-understood location on an ice sheet. But a stress-dependent sliding law is lot better than assuming God told you the distribution of basal velocity (compare Calvo et al (2002)).*]

# *The latest ... partial results ...*

*Theorem I*. Let $n > 1$. Suppose the bed is flat ($b = b_0$). Assume $a$, $\mathbf{U}_b$ satisfy the assumptions. Then

- there exists a $u$ solving the $u$-OBSTACLE PROBLEM,
- it is unique, and
- it is bounded in norm by a constant which depends continuously on the data (i.e. $n$, $\Gamma$, the bound for $a$, the Lipschitz constant for $\mathbf{U}_b$).

*Theorem II*. Let $n > 1$. Assume $b$, $a$, $\mathbf{U}_b$ satisfy the assumptions. Then

- there exists a $u$ solving the $u$-OBSTACLE PROBLEM, and
-
- as long as the bed is not too steep, the solution $u$ is bounded in norm by a constant which depends continuously on the data (i.e. $n$, $\Gamma$, the bound for $a$, the Lipschitz constant for $\mathbf{U}_b$, and a bound for $\nabla b$).

## *Why the blank?*

Theorem II on the last slide depends on a “fixed point argument” (in infinite dimensions, of course).

The particular theorem is the Schaefer fixed point theorem: if $\mathcal{A} : B \to B$ is continuous and compact on Banach spaces, and if $\{w \,\big|\, w = \lambda \mathcal{A}[w] \text{ for some } 0 \le \lambda \le 1\}$ is bounded then $\mathcal{A}$ has a fixed point.

From such an argument one gets a solution, but one doesn’t know if one has more than one. One has a uniqueness problem.

I don’t know if the solution “should be” unique, so I am *not* going to state a conjecture.

# *One can add a temperature-dependent flow law!*

[*In case you want to be revolted further,*]

The $u/T$-OBSTACLE PROBLEM: Suppose

$$\dot{\epsilon}_{ij} = A(T)\sigma^{n-1}\sigma_{ij}$$

[*If you don't know what I mean you won't care what follows anyway . . .*] Given a fixed temperature field $T(x,y,z)$, and given $a$, $b$, and $\mathbf{U}_b$ as before, find $u = H^m$ so that

$$\int_\Omega \left[\mu(u,T)|\nabla u - \Phi_b(u)|^{p-2}(\nabla u - \Phi_b(u)) - u^{1/m}U_b\right] \cdot \nabla(w-u) \geq \int_\Omega a(w-u)$$

for all $w \in \mathcal{K}_0$, where $\mathcal{K}_0 = \left\{w \in W_0^{1,p}(\Omega) \,\middle|\, w \geq 0\right\}$ as before, $\Phi_b(u) = -m\, u^r \nabla b$, where $r = (n+2)/(2n+2)$, as before, and

$$\mu(u,T) = 2(\rho g)^n\, u^{-rn} \int_0^H A(T(s+b))\,(H-s)^{n+1}\, ds.$$

One can probably say a lot about this case, but only a specialist would care . . .

# *Handling time-dependence: the flippant answer*

An approximate solution to the “parabolic”

$$u_t = N(u)$$

is the sequence solving

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\left(N(u^n) + N(u^{n+1})\right).$$

That is, the next one in the sequence solves

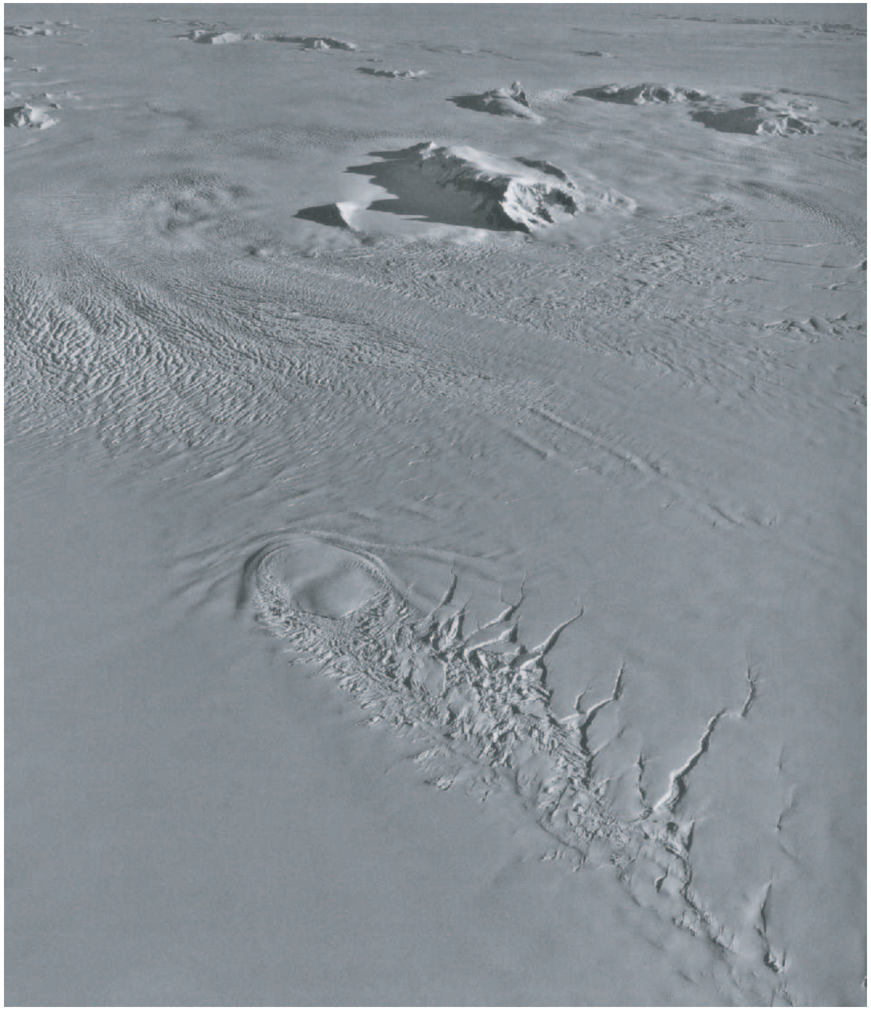
$$u^{n+1} - \frac{\Delta t}{2} N(u^{n+1}) = (\text{known}),$$

which is “elliptic.”

So to solve a time-dependent ice sheet problem one can always semi-discretize and deal with a sequence of steady-state problems.

THE NEXT ONE IS THE LAST SLIDE!

# *My favorite slide*



Palmer Land, Antarctica. *Photo 131, Post & LaChapelle 2000.*

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| Ed: *plug for seminar* |