Flowing ice sheets (with a bit of applied math)

Ed Bueler

Dept of Mathematics and Statistics and Geophysical Institute
University of Alaska Fairbanks

21 May, 2014

supported by NASA grant NNX13AM16G

Outline

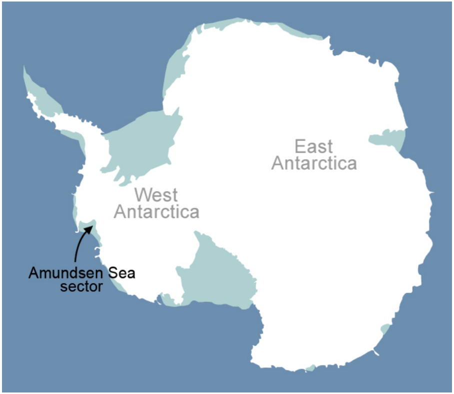
ice sheet flow: an introduction for non-glaciologists

shallow ice approximation for grounded ice sheets

marine ice sheets

news from Antarctica

by Rignot et al. (2014)



[show movie]

Outline

ice sheet flow: an introduction for non-glaciologists

shallow ice approximation for grounded ice sheets

marine ice sheets

ice in glaciers is a viscous fluid

- . . . at least: glaciers are viscous flows at large scales
- *usage*: "ice sheets" are big, shallow glaciers

what is a fluid?

- seriously

what is a fluid?

- seriously
- is it just a collection of particles?

what is a fluid?

- seriously
- is it just a collection of particles?
- a fluid is a mathematical abstraction!

what is a fluid?

- seriously
- is it just a collection of particles?
- a fluid is a mathematical abstraction!
- primary variables:
    - velocity $\mathbf{u}(\mathbf{x}, t)$
    - pressure $p(\mathbf{x}, t)$
    - density $p(\mathbf{x}, t)$

ice in glaciers is a viscous fluid

- liquid water, motor oil, and glacier ice are all modeled as incompressible viscous fluids
- if the glacier ice were a "typical" incompressible viscous fluids like the ocean we would model with Navier-Stokes equations:[1]

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

$$\rho \left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g} \qquad \textit{stress balance}$$

[1] these Navier-Stokes equations in 3D are hard . . . Clay prize

ice is a weird viscous fluid

- . . . but ice is not typical!
- e.g. not relevant in ice sheet flow:
  - turbulence
  - convection
  - coriolis force
  - density-driven flow (weather)

ice is a slow, shear-thinning viscous fluid

- our glacier fluid is
  1. "slow" is a technical term:[2]

$$\rho \left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right) \approx 0 \qquad \iff \qquad \begin{pmatrix}\text{forces of inertia} \\ \text{are negligible}\end{pmatrix}$$

  2. non-Newtonian (shear-thinning):

viscosity ν is not constant

- for "shear-thinning" there is a power law ("Glen law"):

$$(\text{strain rate}) = A(\text{shear stress})^n$$

where $A > 0$ is the ice "softness"

- $1.8 < n < 4.0$? when in doubt: $n = 3$

[2] $Fr \approx 10^{-15}$. Regarding coriolis: $Fr/Ro \approx 10^{-8}$.

ice is a slow, shear-thinning viscous fluid

- notation:
  - τ_{ij} is deviatoric stress tensor
  - $\mathbf{D}u_{ij}$ is strain rate tensor
- the standard ice flow model is Glen-law Stokes:

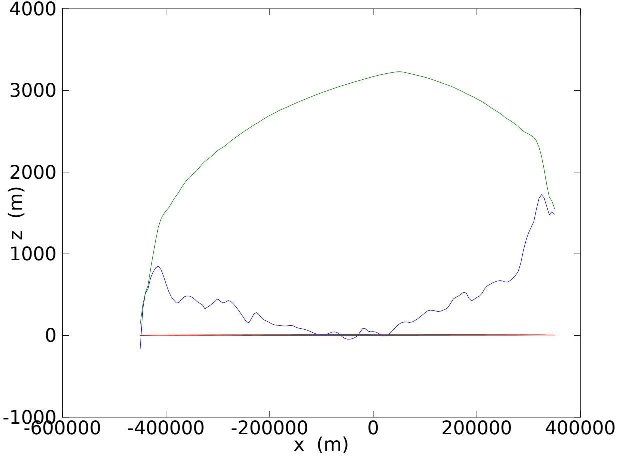
$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

$$0 = -\nabla p + \nabla \cdot \tau_{ij} + \rho \mathbf{g} \qquad \textit{slow stress balance}$$

$$\mathbf{D}u_{ij} = A \left|\tau_{ij}\right|^2 \tau_{ij} \qquad \textit{Glen flow law}$$

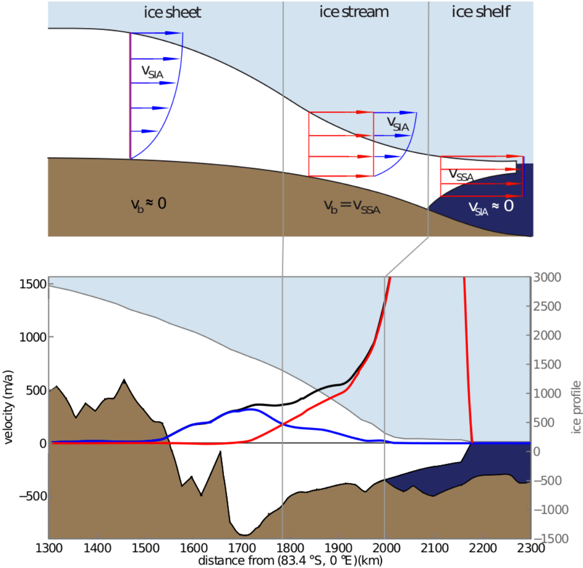
ice sheets are shallow

- cross section of Greenland ice sheet at 71° N
  - green and blue: vertically-exaggerated version
  - in red: without vertical exaggeration



sheets versus streams versus shelves

- non-sliding portions of ice sheets flow by shear deformation
- ice streams slide: contact slip
- "ice shelves" are floating thick ice
- ice shelves flow by extension
  - "membrane" or "plug" flow



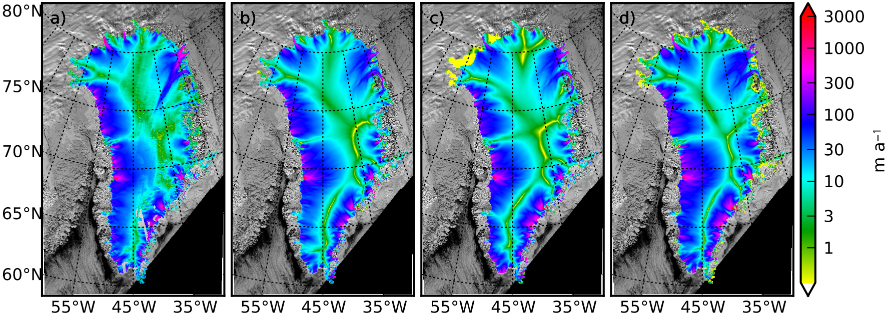
[Lambert glacier and Amery ice shelf, Antarctic]

outstanding viscous flows

- ice sheets have four outstanding properties *as viscous flows*:
    1. slow
    2. shear-thinning
    3. shallow
    4. contact slip

flow results from PISM

- PISM = Parallel Ice Sheet Model (`pism-docs.org`)
- below are 2 km grid results for Greenland; everything evolves; only showing surface velocities
- movement of ice can be measure from space (!)

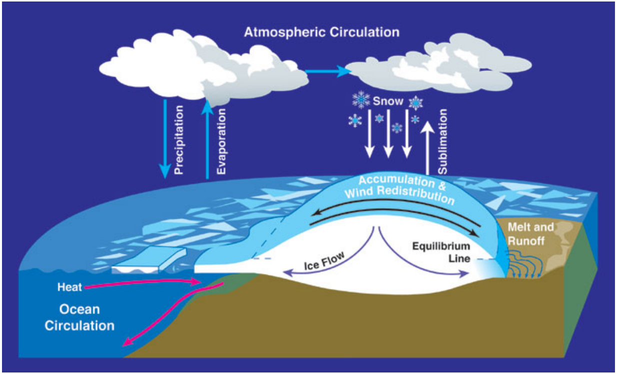


a) observed; b) constant-climate; c) paleo-climate; d) flux-corrected

(computations by Andy Aschwanden)

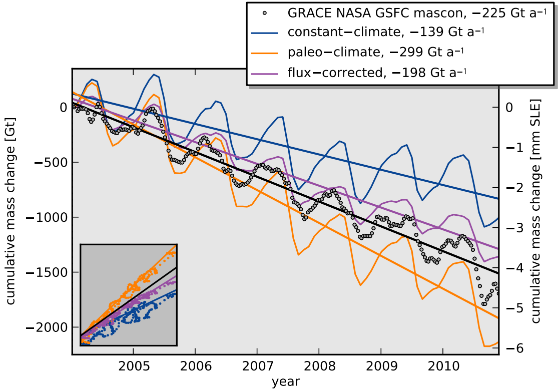
big picture: ice sheet flow interacts with climate

- *mass and energy inputs*: (1) snow adds, (2) sun heats, (3) ocean heats, (4) earth heats
- *mass outputs*: (1) surface meltwater, (2) basal meltwater, (3) ice discharge



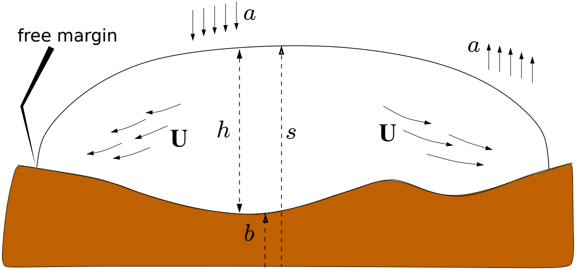
big picture: ice sheet changes over time

- ice sheet mass can be measured from space (!)
- below: curves are (PISM) $+$ (climate model) results
- Greenland ice sheet mass is 2.7×10^9 Gt $\quad (\approx$ km$^3)$
- if *all* Greenland ice melts: 7 m of sea level rise
- if *all* Antarctic ice melts: 61 m of sea level rise



Outline

the main variables

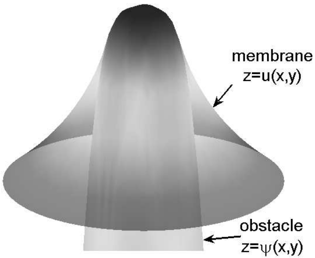


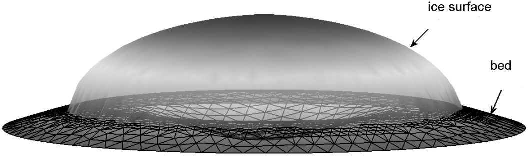
- s = ice surface elevation
- b = bedrock elevation
- h = ice thickness = $s - b$
- $\mathbf{U}$ = horizontal velocity field
- a = surface mass balance (accumulation)

obvious idea: ice surface s is always above the bedrock b

ice sheets: a mathematical modeling analogy

- ice sheet surface = membrane
- bedrock = obstacle





shallow ice approximation (SIA)

- SIA = lubrication approximation of Glen-law Stokes model earlier
- good approximation when:
  - sliding is small or zero
  - bedrock slope is modest
- derive SIA equations by scaling Stokes:
  - $[h]$ is a typical thickness scale
  - $[x]$ is a typical width scale
  - small parameter is $\epsilon = [h]/[x]$

SIA: velocity

- horizontal ice velocity is given by:

$$\mathbf{U} = -\frac{2A}{4}(\rho g)^3 \left[(s-b)^4 - (s-z)^4\right] |\nabla s|^2 \nabla s$$

- no PDE needs to be solved to compute velocity!

SIA: steady state

- mass conservation in steady state:

$$\nabla \cdot \left(\int_b^s \mathbf{U} \, dz \right) = a$$

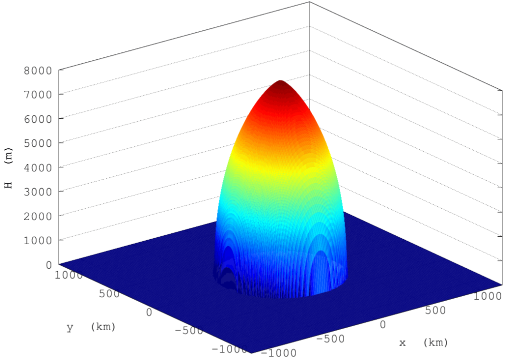
- shallow ice approximation + (steady) mass conservation:

$$-\nabla \cdot \left(\Gamma (s - b)^5 |\nabla s|^2 \nabla s \right) = a$$

  - this is the major SIA equation (. . . a PDE?)
  - computes ice surface s
  - constant $\Gamma > 0$ combines ρ, g, A
  - coefficient $(s - b)^5 \to 0$ at margins

movie of time-dependent SIA

- at right is the Halfar similarity solution
- an exact, time-dependent, zero mass balance solution where the $t \to 0^+$ limit is a delta function
- compare Barenblatt solution of porous medium equation



frames from $t = 4$ months to $t = 10^6$ years, equal spaced in *exponential* time

Outline

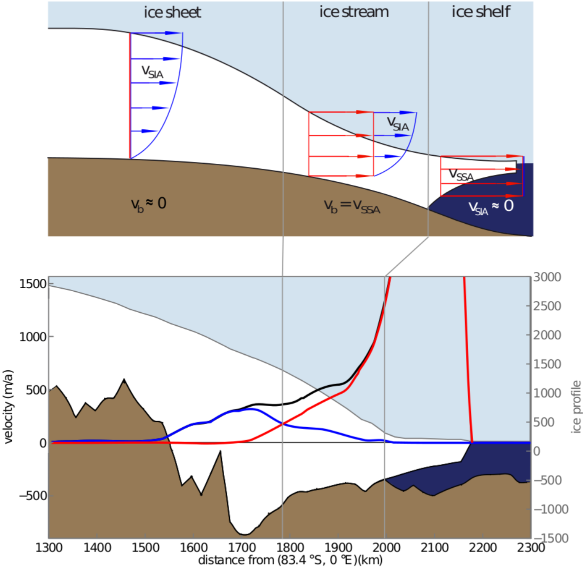
ice sheet flow: an introduction for non-glaciologists

shallow ice approximation for grounded ice sheets

marine ice sheets

sheets versus streams versus shelves

- non-sliding portions of ice sheets flow by shear deformation
- ice streams slide: contact slip
- "ice shelves" are floating thick ice
- ice shelves flow by extension
    - "membrane" or "plug" flow



[Lambert glacier and Amery ice shelf, Antarctic]

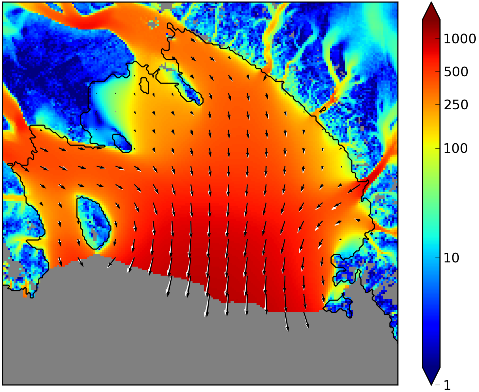
ice shelf versus sea ice

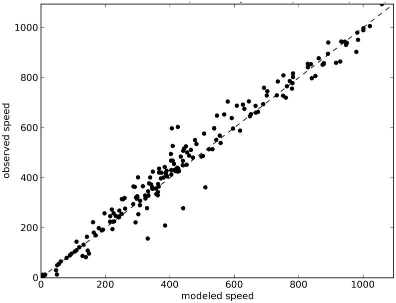
[ice shelf at Thwaites Glacier, Antarctic]

models of ice shelves: they work

- Ross ice shelf (Antarctica) velocity below
  - observed versus computed by SSA model in PISM
  - tuned: single, constant A

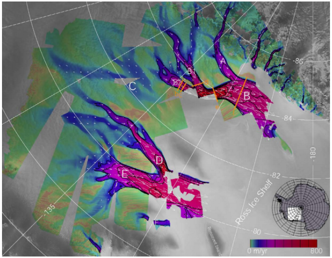
ice velocity (m/year); white=observed, black=model

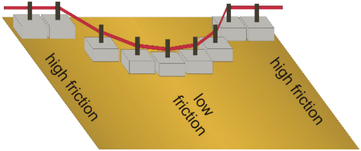




ice streams: an analogy

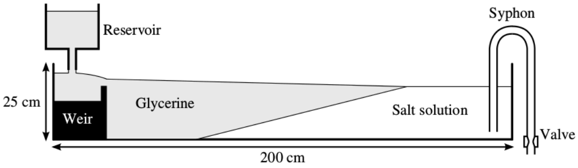
- ice shelves have zero basal resistance
- ice streams emerge where basal resistance is low enough
- basal resistance is low if there is pressurized liquid water underneath the ice sheet
- ice sheet is a membrane which connect sliding ice to upstream and/or lateral non-sliding ice





moving grounding line in the lab

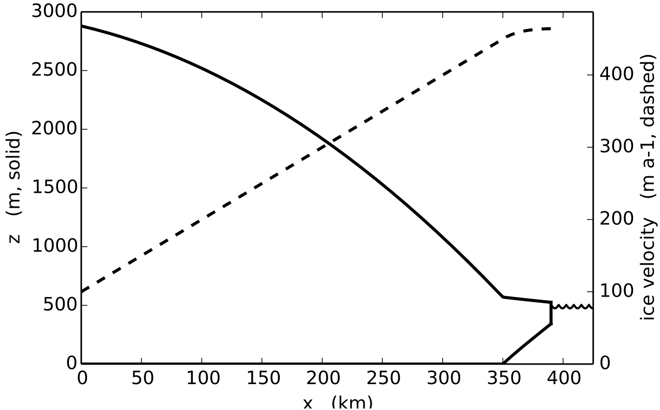
by Pegler et al. (2014)



[show movie]

a steady grounding line from the equations

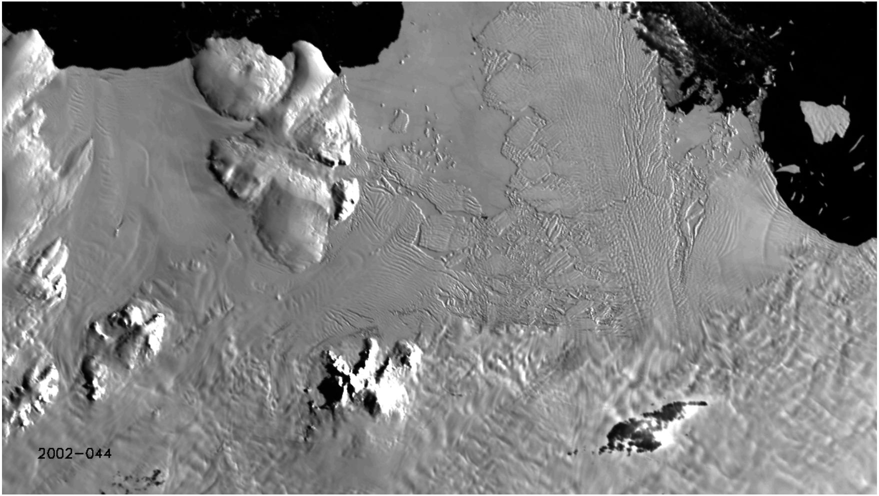
by Bueler (2014) submitted



- thickness (solid) and velocity (dashed) exactly solve the coupled equations for conservation of mass and momentum
- . . . but I'll leave those equations unstated

conclusion

thanks for listening!



Thwaites glacier area, West Antarctica, from February 2002 to February 2014 (Rignot et al. 2014)